Scaling algebras and short-distance analysis for charge carrying quantum fields

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Abstract. The method of scaling algebras, which has been introduced earlier as a means for analyzing the short-distance behaviour of quantum field theories in the setting of the model-independent, operator algebraic approach, is extended to the case of fields carrying superselection charges. A criterion for the preservance of superselection charges in the short-distance scaling limit is proposed. Consequences of this preservance of superselection charges are studied. The conjugate charge of a preserved charge is also preserved, and the preservance of all charges of a quantum field theory in the scaling limit leads to equivalence of local and global intertwiners between superselection sectors.

1 Introduction

In an attempt to analyze the short-distance behaviour of quantum field theories in a completely model-independent manner, and to have a counterpart of renormalization group analysis at short length scales in the setting of general quantum field theory, so-called "scaling algebras" have been introduced some time ago [5]. The idea of this approach is to associate to a given quantum field theory described in terms of local observable algebras [12, 11] a "scaling algebra" of functions depending on a scaling parameter $\lambda > 0$ and taking values in the local observable algebras. These functions are required to have certain properties regarding their localization and energy behaviour as λ tends to zero; roughly

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speaking, the values of the functions at scale parameter λ should be observables localized in spacetime regions of extension proportional to λ, and having energy-momentum transfer proportional to λ^{-1}.

The collection of all these functions, i.e. of all the members of the scaling algebra, may hence be viewed as "orbits" of elements in the local observable algebras under all possible renormalization group transformations. By studying the vacuum expectation values of these functions in the limit $\lambda \to 0$ (the "scaling limit"), one can then analyze the extreme short distance properties of the given quantum field theory.

This programme, initiated in [5], has been further developed in [4, 6, 3, 15]. It leads to a general classification of the short distance behaviour of the given theory which corresponds to the one known in perturbation theory where one distinguishes theories with stable ultraviolet fixed points under renormalization group transformations, as opposed to others with unstable fixed points or no fixed points at all [5].

Moreover, it permits to give a criterion as to when a given quantum field theory possesses "confined charges" which are only visible in the extreme short distance limit while they are absent at finite scale, like the colour charge in QCD [2, 4]. According to this criterion a charge is confined if it arises as a superselection charge in the scaling limit theory of the observables which is not a scaling limit of the superselection charges of the original theory at scale $\lambda = 1$ (see Sec. 5 for discussion). The effectiveness of this criterion has been illustrated in the example of the two-dimensional Schwinger model [4, 6].

However, with the exception of the announcement [16], the scaling algebra method has up to now only been applied in the setting of local observable algebras, not in the context of local field algebras containing charge-carrying local field algebras. In other words, this method has not yet been applied to studying the short-distance behaviour of superselection charges (see [20, 11] and references cited there) and their corresponding charge-carrying fields.

In the present work, we generalize the "scaling algebra" framework in the setting of algebraic quantum field theory in the presence of local field operators transforming non-trivially under the action of a (global) compact gauge group. We also assume that the translations act on the local algebras of field operators, and that there is a translation-invariant vacuum. This then amounts to considering all translation covariant superselection sectors of strictly localizable charges with finite statistics [8]. Our principal interest lies in the behaviour of the superselection charges in the scaling limit.

We propose a criterion specifying what it means that a charge superselection sector of the given quantum field theory is "preserved" in the scaling limit. (Our criterion is, in fact, very similar to the one recently suggested by Morsella [16].) Then we will show that under quite general conditions, a superselection charge is preserved in the scaling limit exactly if this is also the case for the corresponding conjugate charge. As a further application, we extend an earlier result by Roberts [19] (which was obtained for dilation covariant quantum field theories) by showing that in a quantum field theory where all charges are preserved in the scaling limit, the sets of local and global intertwiners for the superselection charges coincide (see the first part of Sec. 4 for explanation of this terminology). This amounts to saying that part of the superselection structure is determined locally if the superselection charges are ultraviolet stable in the sense of being preserved in the scaling limit. Such a property is of some relevance in the construction of superselection theory in a generally covariant setting as recently developed in [23].

This article is organized as follows. In Sec. 2 we define the quantum field theories that we will be considering more precisely. We introduce a class of theories which we call "quantum field theories with gauge group action", abbreviated QFTGA, in the operator-algebraic setting. This class of theories is slightly more general than the class of theories obtained via the Doplicher-Roberts reconstruction from strictly localizable superselection charges (which will be considered in Sec. 4). We introduce the scaling algebra for such QFTGAs, and, in close analogy to [5], we introduce scaling limit states and scaling limit theories and study their basic properties.

Then, in Sec. 3, we consider QFTGAs with more structure, mainly with additional Poincaré covariance and clustering properties, and study what additional properties ensue in the scaling limit.

In Sec. 4 we introduce "quantum field systems with gauge symmetry" (QFSGSs) according to [8]. These are more special QFTGAs which arise by the Doplicher-Roberts reconstruction theorem from the covariant, strictly localizable superselection sectors with finite statistics belonging to a quantum field theory of local observables (cf. again [8]). Charges of this kind would, e.g., correspond to the flavour charges of strong interactions. The reason why we make a distinction between QFTGA and QFSGS is that the scaling limit theories of a QFTGA are again of this type, i.e. are QFTGAs. But scaling limit theories of a QFSGS have in general only the structure of a QFTGA. We summarize parts of the terminology of the theory of superselection sectors and the result on the existence of a corresponding QFSGS, emphasizing the role played by the "field multiplets" in the local field algebras corresponding to each superselection charge.

We will make use of this in Sec. 5, where we will state our criterion of preservance of a charge in the scaling limit in terms of such field multiplets: Our criterion demands that a charge is preserved in the scaling limit if scaled families of such multiplets ("scaled multiplets") have a certain limiting behaviour in the scaling limit. Then we briefly discuss mechanisms for the disappearance of charges in the scaling limit. Quite generally, a charge may disappear in the scaling limit if it takes typically more energy than proportional to λ^{-1} to create the charge within a spacetime region of extension proportional to λ. Moreover, we present some further results on the structure of superselection charges preserved in the scaling limit, like the preservance of the conjugate charge.

In Sec. 6 we state and prove our result on the equivalence of local and global intertwiners if all charges are preserved in the scaling limit.

Shortly before this article was completed, we received a new work by Morsella [17] containing related material.

2 Quantum field theories with gauge group action and their scaling algebras and scaling limits

In the present section we investigate an extension of the "scaling algebra" approach of [5] to quantum field theories that include a structure which we will call a *normal, covariant quantum field theory with gauge group action* (QFTGA) since we will see that this structure has a counterpart in the scaling limit. In the next section we add a few more assumptions, such as Lorentz covariance, geometric modular action, and clustering, but it is not before Section 4 that we introduce a *normal, covariant quantum field system with gauge symmetry* according to [8] which connects quantum field algebras and superselection sectors, and explore some properties of the scaling limits for such theories.

Notation. In the following, we consider quantum field theories on n-dimensional Minkowski-spacetime ($n \geq 2$), which will be identified with $\mathbb{R}^n$, equipped with the Lorentzian metric $\eta = (\eta_{\mu\nu}) = \mathrm{diag}(1, -1, -1, \dots, -1)$. We recall that the set $V_+ := \{(y^0, \dots, y^{n-1}) \in \mathbb{R}^n : (y^0)^2 > (y^1)^2 + \dots + (y^{n-1})^2,\ y^0 > 0\}$ denotes the open forward lightcone and $\overline{V_+}$ its closure. A *double cone* is any set in $\mathbb{R}^n$ of the form $O = x + V_+ \cap y - V_+$ for any pair of $x, y \in \mathbb{R}^n$ so that $y \in x + V_+$. The set of all double cones in $\mathbb{R}^n$ will be denoted generically by $\mathcal{K}$.

Definition 2.1 A quintuple $(\mathcal{F}, \mathscr{U}(\mathbb{R}^n), U(G), \Omega, k)$ is called a *normal, covariant quantum field theory with gauge group action* (QFTGA) if the following properties are fulfilled:

(QFTGA.1) There is a Hilbert-space $\mathcal{H}$ and a family $\{\mathcal{F}(O)\}_{O \in \mathcal{K}}$ of von Neumann algebras on $\mathcal{H}$ which is indexed by the members O of the set $\mathcal{K}$ of all double cones in n-dimensional Minkowski spacetime. It will be assumed that isotony holds, i.e.

$$O_1 \subset O \Rightarrow \mathcal{F}(O_1) \subset \mathcal{F}(O)\,.$$

Hence, one may form the smallest C^*-algebra $\mathfrak{F} := \overline{\bigcup_O \mathcal{F}(O)}^{C^*}$ in $B(\mathcal{H})$ containing all local field algebras $\mathcal{F}(O)$. (In the above quintuple, $\mathcal{F}$ is short for the family $\{\mathcal{F}(O)\}_{O \in \mathcal{K}}$.)

(QFTGA.2) There is a strongly continuous unitary representation $\mathbb{R}^n \ni a \mapsto \mathscr{U}(a) \in B(\mathcal{H})$ of the group of translations $\mathbb{R}^n$ on $\mathcal{H}$ whose action on $\{\mathcal{F}(O)\}_{O \in \mathcal{K}}$ is covariant, i.e.

$$\mathscr{U}(a)\mathcal{F}(O)\mathscr{U}(a)^* = \mathcal{F}(O + a)\,, \quad a \in \mathbb{R}^n\,,\ O \in \mathcal{K}\,.$$

Moreover, it will be assumed that the relativistic spectrum condition holds: The joint spectrum of the selfadjoint generators of $\mathscr{U}(\mathbb{R}^n)$ is cointained in the closed forward lightcone $\overline{V^+}$.

(QFTGA.3) There is a compact group G, and a strongly continuous,[2] faithful representation $G \ni g \mapsto U(g) \in B(\mathcal{H})$ of the group G on $\mathcal{H}$. It is assumed that the action of this unitary representation on $\{\mathcal{F}(O)\}_{O \in \mathcal{K}}$ preserves localization, i.e.

$$U(g)\mathcal{F}(O)U(g)^* = \mathcal{F}(O)\,, \quad g \in G\,,\ O \in \mathcal{K}\,,$$

and also that this group representation commutes with the translations:

$$U(g)\mathscr{U}(a) = \mathscr{U}(a)U(g)\,, \quad g \in G\,,\ a \in \mathbb{R}^n\,.$$

G will be called the *gauge group.*

[2] whenever this makes sense, i.e. when G possesses continuous parts

(QFTGA.4) There is a unit vector $\Omega \in \mathcal{H}$ which is invariant under all $\mathscr{U}(a)$, $a \in \mathbb{R}^n$, and under all $U(g)$, $g \in G$, and which moreover has the cyclicity property $\overline{\mathfrak{F}\Omega} = \mathcal{H}$. This vector is called the *vacuum vector*.

(QFTGA.5) There is an element k contained in the centre of G and fulfilling $k^2 = 1_G$ (the unit group element) so that, upon setting

$$F_\pm := \frac{1}{2}(F \pm U(k)FU(k)^*) ,$$

the following relations hold whenever $F \in \mathcal{F}(O_1)$, $F' \in \mathcal{F}(O_2)$, and the double cones O_1 and O_2 are spacelike sparated:

$$F_+F'_+ = F'_+F_+ , \quad F_+F'_- = F'_-F_+ , \quad F_-F'_- = -F'_-F_- . \tag{2.1}$$

These properties are referred to as *normal commutation relations.*

Remark. It was already mentioned in the introduction that the definition of a QFTGA is slightly more general than that of a quantum field system with gauge symmetry (see Sec. 4) which is more directly related to the theory of superselection charges; however, the differences are minute and mainly of technical nature. The advantage of working with QFTGAs is that their structure is stable with respect to passing to scaling limit theories, as will become clear in the present section.

The next task is to introduce the counterpart of the scaling algebra for a QFTGA which was defined in [5] for quantum field theories formulated in terms of local observable algebras. To that end, we assume that we are given an arbitrary normal, covariant quantum field theory with gauge group action $(\mathcal{F}, \mathscr{U}(\mathbb{R}^n), U(G), \Omega, k)$ (henceforth called the "underlying QFTGA") and keep it fixed. It will be convenient to introduce the following notation for the adjoint actions of translations and gauge group:

$$\alpha_a(F) := \mathscr{U}(a)F\mathscr{U}(a)^* , \quad \beta_g(F) := U(g)FU(g)^* ,$$

for all $F \in \mathfrak{F}$, $a \in \mathbb{R}^n$, $g \in G$.

Definition 2.2 For each $O \in \mathcal{K}$, we define $\underline{\mathfrak{F}}(O)$ as the set of all functions $\underline{F} : \mathbb{R}^+ \to \mathfrak{F}$, $\lambda \mapsto \underline{F}_\lambda$, having the following properties:

(a) $\underline{F}_\lambda \in \mathcal{F}(\lambda O)$,

(b) $||\,\underline{F}\,|| := \sup_\lambda ||\underline{F}_\lambda|| < \infty$,

(c) $||\,\underline{\alpha}_a(\underline{F}) - \underline{F}\,|| \to 0$ as $a \to 0$, where

$$(\underline{\alpha}_a(\underline{F}))_\lambda := \alpha_{\lambda a}(\underline{F}_\lambda) , \tag{2.2}$$

(d) $||\,\underline{\beta}_g(\underline{F}) - \underline{F}\,|| \to 0$ as $g \to 1_G$, where

$$(\underline{\beta}_g(\underline{F}))_\lambda := \beta_g(\underline{F}_\lambda) . \tag{2.3}$$

In [5] the case was considered that $\mathfrak{F}$ is an observable algebra. In that case, the action of the gauge group $U(G)$ on $\mathcal{H}$ is trivial, and spacelike commutativity holds for the local algebras $\mathcal{F}(O)$, meaning that $\mathcal{F}(O_1) \subset \mathcal{F}(O_2)'$ if O_1 and O_2 are spacelike separated. The motivation for imposing the conditions (a-d) above is similar as for the scaling algebra in the case that $\mathcal{F}$ is an observable algebra discussed in [5]. The idea is to view the $\underline{F}_\lambda$ as the image of an element $F \in \mathfrak{F}$ under the action of any "renormalization group transformation" R_λ (so one should think of $\underline{F}_\lambda$ as $R_\lambda(F)$). In other words, the collection of all functions $\lambda \mapsto \underline{F}_\lambda$ with the above stated properties corresponds to all possible orbits of elements in $\mathfrak{F}$ under all (abstract) renormalization group transformations. The general properties of renormalization group transformations in the present, model-independent setting are hence encoded by the conditions (a-d). We point out that (c) ensures that the energy-momentum transferred by $\underline{F}_\lambda$ scales like const.$\cdot 1/\lambda$, see [5] for further discussion.

As has been indicated to us by D. Buchholz, it should be noted that there may actually be situations where the lifted action of the gauge transformations ought to be defined differently than in (2.3). This occurs for example if the charges of the theory have a dimension which isn't independent of length or energy (in this sense, they are "dimensionful" charges), and this can happen in two-dimensional models. For the time being, we neglect this possibility, but we point out that it deserves attention.

There are some simple consequences of Def. 2.2 which we briefly put on record here, see [5] for more details. First, it is easy to see that each $\underline{\mathfrak{F}}(O)$, $O \in \mathcal{K}$, is a C^*-algebra with respect to the C^*-norm introduced in (b) when the algebraic operations are defined pointwise for each λ. Clearly one also has isotony,

$$O_1 \subset O \Rightarrow \underline{\mathfrak{F}}(O_1) \subset \underline{\mathfrak{F}}(O)\,.$$

One can thus form the C^*-algebra $\underline{\mathfrak{F}} = \overline{\bigcup_O \underline{\mathfrak{F}}(O)}^{C^*}$. The "lifted" actions $\underline{\alpha}_{\mathbb{R}^n}$ and $\underline{\beta}_G$ of translations and gauge group, defined in (2.2) and (2.3), respectively, act by automorphisms on $\underline{\mathfrak{F}}$ under preservation of the corresponding covariance properties, i.e.

$$\underline{\alpha}_a(\underline{\mathfrak{F}}(O)) = \underline{\mathfrak{F}}(O+a)\,, \quad \underline{\beta}_g(\underline{\mathfrak{F}}(O)) = \underline{\mathfrak{F}}(O)\,. \tag{2.4}$$

Moreover, we may define

$$\underline{F}_\pm = \frac{1}{2}(\underline{F} \pm \underline{\beta}_k(\underline{F}))$$

and hence obtain relations similar to (2.1) for $\underline{F} \in \underline{\mathfrak{F}}(O_1)$, $\underline{F}' \in \underline{\mathfrak{F}}(O_2)$ and O_1 and O_2 spacelike separated. Finally we note that one may demonstrate the existence of a wealth of elements in $\underline{\mathfrak{F}}$ as follows. Let μ be a left-invariant Borel-measure on G and let h be any continuous, compactly supported function on $\mathbb{R}^d \times G$. Pick any uniformly bounded function $\mathbb{R}^+ \ni \lambda \mapsto X_\lambda \in \mathfrak{F}$ so that $X_\lambda \in \mathcal{F}(\lambda O)$ for each λ and some $O \in \mathcal{K}$, and define

$$\underline{F}_\lambda := \int d^n a\, d\mu(g)\, h(a,g)\, \alpha_{\lambda a}(\beta_g(X_\lambda)) \tag{2.5}$$

where the integral is to be understood in the weak sense. Then it is easily checked that $\mathbb{R}^+ \ni \lambda \mapsto \underline{F}_\lambda$ is contained in $\underline{\mathfrak{F}}(O^\times)$ whenever $O^\times$ is any open neighbourhood of $\overline{O} + \overline{\bigcup_{g\in G} \operatorname{supp} h(\,.\,,g)}$.

Having defined the scaling field algebra $\underline{\mathfrak{F}}$ of the underlying QFTGA, we may associate with any locally normal state ω' on $\mathfrak{F}$ [3] a parametrized family $(\underline{\omega}'_\lambda)_{\lambda>0}$ of states on $\underline{\mathfrak{F}}$, where

$$\underline{\omega}'_\lambda(\underline{F}) := \omega'(\underline{F}_\lambda)\,, \quad \underline{F} \in \underline{\mathfrak{F}}\,.$$

As in [5], we adopt the following definition of scaling limit states.

Definition 2.3 For each locally normal state ω' on $\mathfrak{F}$, we regard the family $(\underline{\omega}'_\lambda)_{\lambda>0}$ as a generalized sequence directed towards $\lambda = 0$. Hence, by the Banach-Alaoglu theorem [18], the family $(\underline{\omega}'_\lambda)_{\lambda>0}$ on the C^*-algebra $\underline{\mathfrak{F}}$ possesses weak-* limit points. This set of weak-* limit points will be denoted by $\{\omega'_{0,\iota} : \iota \in \mathbb{I}\}$ where $\mathbb{I}$ is a suitable index set, or simply by $\mathrm{SL}^{\mathfrak{F}}(\omega')$. Each $\omega'_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega')$ is a state on $\underline{\mathfrak{F}}$, and is called a *scaling limit state of* ω'.

We note that the definition of weak-* limit points means that there exists for each label ι a directed set K_ι together with a generalized sequence $(\lambda^{(\iota)}_\kappa)_{\kappa\in K_\iota}$ of positive numbers converging to 0 so that

$$\omega'_{0,\iota}(\underline{F}) = \lim_\kappa \underline{\omega}'_{\lambda^{(\iota)}_\kappa}(\underline{F})\,, \quad \underline{F} \in \underline{\mathfrak{F}}\,.$$

Again following [5], we introduce for each scaling limit state $\omega'_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega')$ its GNS-representation $(\pi_{0,\iota}, \mathcal{H}_{0,\iota}, \Omega_{0,\iota})$ and define

$$\mathcal{F}_{0,\iota}(O) := \pi_{0,\iota}(\underline{\mathfrak{F}}(O))''\,, \quad \mathfrak{F}_{0,\iota} := \overline{\bigcup_O \mathcal{F}_{0,\iota}(O)}^{C^*}\,.$$

Many of the following results (containing also some new definitions) concerning the structure of scaling limit states and their associated GNS-representations in the present setting are generalizations of similar statements in [5].

Proposition 2.4 *1. For each pair of locally normal states ω' and ω'' on $\mathfrak{F}$ it holds that*

$$\mathrm{SL}^{\mathfrak{F}}(\omega') = \mathrm{SL}^{\mathfrak{F}}(\omega'')\,.$$

2. *Let ω' be a locally normal state on $\mathfrak{F}$. Then each $\omega'_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega')$ is invariant under the actions of $\underline{\alpha}_a$, $a \in \mathbb{R}^n$, and $\underline{\beta}_g$, $g \in G$:*

$$\omega'_{0,\iota} \circ \underline{\alpha}_a = \omega'_{0,\iota}\,, \quad \omega'_{0,\iota} \circ \underline{\beta}_g = \omega'_{0,\iota}\,.$$

Hence, there are unitary group representations of the translation group and the gauge group on $\mathcal{H}_{0,\iota}$ which are, respectively, defined by

$$\mathscr{U}_{0,\iota}(a)\pi_{0,\iota}(\underline{F})\Omega_{0,\iota} := \pi_{0,\iota}(\underline{\alpha}_a(\underline{F}))\Omega_{0,\iota}\,, \quad U_{0,\iota}(g)\pi_{0,\iota}(\underline{F})\Omega_{0,\iota} := \pi_{0,\iota}(\underline{\beta}_g(\underline{F}))\Omega_{0,\iota}$$

for all $a \in \mathbb{R}^n$, $g \in G$, and $\underline{F} \in \underline{\mathfrak{F}}$.

[3] a state ω' on $\mathfrak{F}$ is called *locally normal* if $\omega' \restriction \mathcal{F}(O)$ is normal for each $O \in \mathcal{K}$

3. *The unitary group representations* $\mathscr{U}_{0,\iota}(a)$, $a \in \mathbb{R}^n$, *and* $U_{0,\iota}(g)$, $g \in G$, *are continuous and have the properties*

$$\mathscr{U}_{0,\iota}(a)\mathcal{F}_{0,\iota}(O)\mathscr{U}_{0,\iota}(a)^* = \mathcal{F}_{0,\iota}(O+a)\,, \quad U_{0,\iota}(g)\mathcal{F}_{0,\iota}(O)U_{0,\iota}(g)^* = \mathcal{F}_{0,\iota}(O)$$

for all $a \in \mathbb{R}^n$, $g \in G$ *and* $O \in \mathcal{K}$. *Moreover, the unitary translation group* $\mathscr{U}_{0,\iota}(a)$, $a \in \mathbb{R}^n$, *fulfills the relativistic spectrum condition.*

4. *The set* $N_{0,\iota}$ *of all* $g \in G$ *so that* $U_{0,\iota}(g)\psi = \psi$ *holds for all* $\psi \in \mathcal{H}_{0,\iota}$ *is a closed normal subgroup of* G. *Therefore,*

$$U^\bullet_{0,\iota} : G^\bullet_{0,\iota} \ni g^\bullet \mapsto U_{0,\iota}(g) \tag{2.6}$$

is a continuous faithful representation of the factor group $G^\bullet_{0,\iota} = G/N_{0,\iota}$. *Here,* $g \mapsto g^\bullet \equiv g^\bullet_{0,\iota}$ *is the quotient map, and in* (2.6), g *is any element in the pre-image of* $g^\bullet$ *with respect to the quotient map.*

5. *Define for* $\boldsymbol{f} \in \mathfrak{F}_{0,\iota}$,

$$\boldsymbol{f}_\pm := \frac{1}{2}(\boldsymbol{f} \pm U^\bullet_{0,\iota}(k^\bullet)\boldsymbol{f}U^\bullet_{0,\iota}(k^\bullet)^*)$$

where $k^\bullet = k^\bullet_{0,\iota}$. *Then the following holds: If* O_1 *and* O_2 *are spacelike separated double cones and* $\boldsymbol{f} \in \mathcal{F}_{0,\iota}(O_1)$, $\boldsymbol{f}' \in \mathcal{F}_{0,\iota}(O_2)$, *one has the relations*

$$\boldsymbol{f}_+\boldsymbol{f}'_+ = \boldsymbol{f}'_+\boldsymbol{f}_+\,, \quad \boldsymbol{f}_+\boldsymbol{f}'_- = \boldsymbol{f}'_-\boldsymbol{f}_+\,, \quad \boldsymbol{f}_-\boldsymbol{f}'_- = -\boldsymbol{f}'_-\boldsymbol{f}_-\,. \tag{2.7}$$

6. *The previous statements yield the following corollary: Let* ω' *be a locally normal on* $\mathfrak{F}$ *(of the underlying QFTGA) and* $\omega'_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega')$ *an arbitrary scaling limit state, then the corresponding scaling limit objects* $(\mathcal{F}_{0,\iota}, \mathscr{U}_{0,\iota}(\mathbb{R}^n), U^\bullet_{0,\iota}(G^\bullet_{0,\iota}), \Omega_{0,\iota}, k^\bullet_{0,\iota})$ *form again a normal, covariant quantum field theory with gauge group action (which will be called a* scaling limit QFTGA *of the underlying QFTGA corresponding to* $\omega'_{0,\iota}$*).*

Proof. Ad 1. The proof is analogous to that in [5], which uses an argument due to Roberts [19] showing that

$$||(\omega' - \omega'') \restriction \mathcal{F}(\lambda O)|| \to 0 \ \text{ as } \ \lambda \to 0 \tag{2.8}$$

holds for any pair of locally normal states ω' and ω'' on $\mathfrak{F}$ and $O \in \mathcal{K}$ as a consequence of

$$\bigcap_{O \ni 0} \mathcal{F}(O) = \mathbb{C} \cdot 1\,.$$

This latter property holds also for the local field algebras owing to the spectrum condition for the translation group and normal commutation relations (2.1), see [5] for details.

Ad 2. The invariance property is obvious for the case that ω' coincides with the vacuum state $\omega(F) = \langle \Omega, F\Omega \rangle$ on $\mathfrak{F}$. Then (2.8) implies the analogous property for any other locally normal state.

Ad 3. The continuity follows simply from assumptions (c) and (d) of Def. 2.2. The covariance properties are implied by (2.4). The spectrum condition for the translations may be proved as in [5].

Ad 4. By construction, $U^{\bullet}_{0,\iota}$ is a faithful unitary representation of $G^{\bullet}_{0,\iota}$ on $\mathcal{H}_{0,\iota}$. Continuity follows since the quotient map $g \mapsto g^{\bullet}$ is open.

Ad 5. As indicated above, the relations (2.1) carry over to the scaling algebra $\underline{\mathfrak{F}}$ by setting $\underline{F}_{\pm} = \frac{1}{2}(\underline{F} \pm \underline{\beta}_k(\underline{F}))$. The corresponding relations for the scaling limit theories follow directly. (It may however happen that $k \in N_{0,\iota}$; in this case, the last, "fermionic" relation of (2.7) is absent, and spacelike commutativity holds for the local scaling limit algebras $\mathcal{F}_{0,\iota}(O)$, $O \in \mathcal{K}$.)

Henceforth, we will (without restriction of generality in view of 1. of Prop. 2.4) always consider scaling limit states $\omega_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega)$ where $\omega(\,.\,) = \langle \Omega, \,.\,\Omega \rangle$ denotes the vacuum state.

As was done in [5], we will identify scaling limit theories which are isomorphic in a sense that we will describe next.

Definition 2.5 Let

$$(\mathcal{F}_{0,\iota}, \mathscr{U}_{0,\iota}(\mathbb{R}^n), U^{\bullet}_{0,\iota}(G^{\bullet}_{0,\iota}), \Omega_{0,\iota}, k^{\bullet}_{0,\iota}) \quad \text{and} \quad (\mathcal{F}_{0,\gamma}, \mathscr{U}_{0,\gamma}(\mathbb{R}^n), U^{\bullet}_{0,\gamma}(G^{\bullet}_{0,\gamma}), \Omega_{0,\gamma}, k^{\bullet}_{0,\gamma})$$

be two scaling limit theories of an underlying QFTGA. These two scaling limit theories will be called *isomorphic* if there exists a C^*-algebraic isomorphism $\phi : \mathfrak{F}_{0,\iota} \to \mathfrak{F}_{0,\gamma}$ so that the following properties hold:

$$\begin{aligned} \phi(\mathcal{F}_{0,\iota}(O)) &= \mathcal{F}_{0,\gamma}(O)\,, \quad O \in \mathcal{K}\,, \\ \phi \circ \mathrm{Ad}\,\mathscr{U}_{0,\iota}(a) &= \mathrm{Ad}\,\mathscr{U}_{0,\gamma}(a) \circ \phi\,, \quad a \in \mathbb{R}^n\,, \\ \phi \circ \mathrm{Ad}\,U_{0,\iota}(g) &= \mathrm{Ad}\,U_{0,\gamma}(g) \circ \phi\,, \quad g \in G\,. \end{aligned}$$

Note that the last property induces a natural identification between $N_{0,\iota}$ and $N_{0,\gamma}$ and hence a natural identification $G^{\bullet}_{0,\iota} \ni g^{\bullet}_{0,\iota} \mapsto g^{\bullet}_{0,\gamma} \in G^{\bullet}_{0,\gamma}$, so that one obtains, in consequence,

$$\phi \circ \mathrm{Ad}\,U^{\bullet}_{0,\iota}(g^{\bullet}_{0,\iota}) = \mathrm{Ad}\,U^{\bullet}_{0,\gamma}(g^{\bullet}_{0,\gamma}) \circ \phi$$

which holds in particular with $k^{\bullet}_{0,\iota}$ and $k^{\bullet}_{0,\gamma}$ inserted for $g^{\bullet}_{0,\iota}$ and $g^{\bullet}_{0,\gamma}$, respectively.

We will moreover say that two isomorphic scaling limit theories have a *unique vacuum structure* if the connecting isomorphism also has the property

$$\omega_{0,\gamma} \circ \phi = \omega_{0,\iota}\,.$$

Following once more [5], one may now classify a given underlying QFTGA according to the following (mutually exclusive) possibilities:

(1) All scaling limit QFTGAs are isomorphic, and $\mathfrak{F}_{0,\iota}$ is non-abelian. Then the underlying QFTGA is said to have a *unique quantum scaling limit.*

(2) All scaling limit QFTGAs are isomorphic, and $\mathfrak{F}_{0,\iota}$ is abelian. In this case one says that the underlying QFTGA has a *classical scaling limit.*

(3) There are scaling limit QFTGAs which are non-isomorphic. One then says that the underlying QFTGA has a *degenerate scaling limit*.

The interpretation of these cases is as in the case of observable algebras [5]; see this reference for further discussion. The first case would correspond to an underlying theory which has a single, stable ultraviolet fixed point. The second case is thought to correspond to an underlying theory which has no ultraviolet fixed point. The third case is in a sense intermediate, the underlying theory has a very irregular behaviour at small scales and has various, most likely unstable, ultraviolet fixed points.

We next put on record a result from [5] connecting the uniqueness of the scaling limit with the existence of a dilation symmetry in the scaling limit theories. The proof proceeds exactly as in the cited reference.

Proposition 2.6 [5] *Assume that all the scaling limit QFTGAs*

$$(\mathcal{F}_{0,\iota}, \mathscr{U}_{0,\iota}(\mathbb{R}^n), U^{\bullet}_{0,\iota}(G^{\bullet}_{0,\iota}), \Omega_{0,\iota}, k^{\bullet}_{0,\iota}), \quad \iota \in \mathbb{I},$$

of the underlying QFTGA are isomorphic, i.e. that we are in case **(1)** *or* **(2)** *of the just given classification. Then for each* $\iota \in \mathbb{I}$ *there exists a family* $(\delta^{(0,\iota)}_\mu)_{\mu>0}$ *of automorphisms of* $\mathfrak{F}_{0,\iota}$ *acting as dilations in the corresponding scaling limit theory, which means that the following relations hold:*

$$\begin{aligned}
\delta^{(0,\iota)}_\mu(\pi_{0,\iota}(\underline{\mathfrak{F}}(O)) &= \pi_{0,\iota}(\underline{\mathfrak{F}}(\mu O)), \quad \mu > 0,\ O \in \mathcal{K}, \\
\delta^{(0,\iota)}_\mu \circ \mathrm{Ad}\, \mathscr{U}_{0,\iota}(a) &= \mathrm{Ad}\, \mathscr{U}_{0,\iota}(\mu a) \circ \delta^{(0,\iota)}_\mu, \quad a \in \mathbb{R}^n,\ \mu > 0, \\
\delta^{(0,\iota)}_\mu \circ \mathrm{Ad}\, U_{0,\iota}(g) &= \mathrm{Ad}\, U_{0,\iota}(g) \circ \delta^{(0,\iota)}_\mu, \quad g \in G,\ \mu > 0.
\end{aligned}$$

Furthermore, if the underlying QFTGA also has a unique vacuum structure in the scaling limit, then it follows that the family of dilations leaves the scaling limit states invariant: $\omega_{0,\iota} \circ \delta^{(0,\iota)}_\mu = \omega_{0,\iota}$, $\iota \in \mathbb{I}$, $\mu > 0$.

3 Scaling limits for QFTGAs with additional properties

In the present section we consider an underlying QFTGA with additional properties, such as Lorentz-covariance, spacelike clustering and geometric modular action, and we will investigate which further properties for the scaling limit theories ensue. More precisely, let $(\mathcal{F}, \mathscr{U}(\mathbb{R}^n), U(G), \Omega, k)$ be the underlying QFTGA, assumed to satisfy the conditions (QFTGA.1-5) of Def. 2.1. We will consider the following additional properties:

(QFTGA.6) (Lorentz covariance) There is a strongly continuous unitary representation $\tilde{\mathcal{L}}^\uparrow_+ \ni L \mapsto \tilde{\mathscr{U}}(L) \in B(\mathcal{H})$ of the covering group of the proper, orthochronous Lorentz group $\mathcal{L}^\uparrow_+$ (in d dimensions) on $\mathcal{H}$ so that the following relations are fulfilled:

$$\begin{aligned}
\tilde{\mathscr{U}}(L)\mathscr{U}(a) &= \mathscr{U}(\Lambda(L)a)\tilde{\mathscr{U}}(L), \\
\tilde{\mathscr{U}}(L)U(g) &= U(g)\tilde{\mathscr{U}}(L), \\
\tilde{\mathscr{U}}(L)\mathcal{F}(O)\tilde{\mathscr{U}}(L)^* &= \mathcal{F}(\Lambda(L)O), \qquad \tilde{\mathscr{U}}(L)\Omega = \Omega
\end{aligned}$$

for all $L \in \tilde{\mathcal{L}}_+^\uparrow$, $a \in \mathbb{R}^n$, $g \in G$ and $O \in \mathcal{K}$, where $\tilde{\mathcal{L}}_+^\uparrow \ni L \mapsto \Lambda(L) \in \mathcal{L}_+^\uparrow$ denotes the covering projection.

(QFTGA.7) (Irreducibility) $\mathfrak{F}' = \mathbb{C} \cdot 1$.

(QFTGA.8) (Spacelike clustering) We will assume that a uniform clustering bound holds on the vacuum (for spacetime dimension $d \geq 3$). To formulate this, we use the following notation. Elements in the $x^0 = 0$ hyperplane will be denoted by $\boldsymbol{x} \in \mathbb{R}^{n-1}$ and identified with $(0, \boldsymbol{x}) \in \mathbb{R}^n$. We define the derivation

$$\partial_0(F) := -i \left. \frac{d}{dx^0} \right|_{x^0=0} \alpha_{(x^0,\mathbf{0})}(F)$$

on the domain $D(\partial_0)$ of all $F \in \mathfrak{F}$ so that the (weak) derivative on the right hand side exists as an element in $\mathfrak{F}$. Note that $D(\partial_0)$ is a weakly dense subset of $\mathfrak{F}$. Then our assumption on the existence of a uniform spacelike clustering bound is: There exists, for the given underlying QFTGA, a constant $c > 0$ so that for each double cone O_r having spherical base of radius r in the $x^0 = 0$ hyperplane there holds the bound

$$|\omega(F_1\alpha_{\boldsymbol{x}}(F_2)) - \omega(F_1)\omega(F_2)| \leq \frac{cr^{n-1}}{|\boldsymbol{x}|^{n-2}}(||F_1||\,||\partial_0(F_2)|| + ||\partial_0(F_1)||\,||F_2||)$$

for all $F_1, F_2 \in \mathcal{F}(O_r) \cap D(\partial_0)$ as soon as $|\boldsymbol{x}| > 3r$.

(QFTGA.9) (Geometric modular action) A *wedge region* is any Lorentz-transformed copy of the so-called right wedge $W_R := \{(x^0, \ldots, x^{n-1}) : |x^1| < x^0,\ x^0 > 0\}$. For this right wedge, we define the wedge-reflection map $r_R : \mathbb{R}^n \to \mathbb{R}^n$ by

$$r_R(x^0, x^1, x^2, \ldots, x^{n-1}) := (-x^0, -x^1, x^2, \ldots, x^{n-1}),$$

and the Lorentz-boosts

$$\begin{aligned} &\Lambda_R(t)(x^0, x^1, x^2, \ldots, x^{n-1}) \\ &:= \ (\cosh(t)x^0 + \sinh(t)x^1, \sinh(t)x^0 + \cosh(t)x^1, x^2, \ldots, x^{n-1}). \end{aligned}$$

For any other wedge-region $W = \Lambda W_R$ with a suitable Lorentz-transformation Λ, we define $r_W := \Lambda j_R \Lambda^{-1}$ and $\Lambda_W(t) := \Lambda\Lambda_R(t)\Lambda^{-1}$.

For each wedge region W in $\mathbb{R}^n$, the vacuum vector Ω of the underlying QFTGA is cyclic and separating for the von Neumann algebra $\mathcal{F}(W) = \{\mathcal{F}(O) : \overline{O} \subset W,\ O \in \mathcal{K}\}''$. Hence, there correspond to each wedge region W the Tomita-Takesaki modular objects J_W, Δ_W associated with $\mathcal{F}(W), \Omega$ [22]. It will then be assumed that, in the presence of (QFTGA.6), these modular objects act geometrically in the following way:

$$\begin{aligned} J_W \tilde{\mathscr{U}}(L) J_W &= \tilde{\mathscr{U}}(\widetilde{\mathrm{Ad} r_W L}), \qquad J_W \mathscr{U}(a) J_W = \mathscr{U}(r_W a), \quad L \in \tilde{\mathcal{L}}_+^\uparrow,\ a \in \mathbb{R}^n, \\ \Delta_W^{it} &= \tilde{\mathscr{U}}(\widetilde{\Lambda_W(2\pi t)}), \quad t \in \mathbb{R}, \\ J_W \mathcal{F}(O) J_W &= \mathcal{F}^t(r_W O), \quad O \in \mathcal{K}. \end{aligned}$$

In these equations, we have denoted by $\widetilde{\mathrm{Ad}j_W}$ the lift of the adjoint action of r_W to $\tilde{\mathcal{L}}^\uparrow_+$, and by $\widetilde{\Lambda_W(t)}$ the lift of $\Lambda_W(t)$ to $\tilde{\mathcal{L}}^\uparrow_+$ (both of which exist, cf. [9]). Moreover, we have introduced the so-called "twisted" local von Neumann algebras

$$\mathcal{F}^t(O) := V\mathcal{F}(O)V^*, \quad O \in \mathcal{K}, \tag{3.1}$$

where the twisting operator V is a unitary on $\mathcal{H}$ defined by

$$V := (1+i)^{-1}(1+U(k)). \tag{3.2}$$

Note that the algebras $\mathcal{F}^t(O_1)$ and $\mathcal{F}^t(O_2)$ commute for spacelike separated O_1 and O_2 on account of the assumed normal commutation relations.

We shall continue our investigation of the scaling limit theories of an underlying QFTGA satisfying some, or all, of the just stated additional conditions. In order to do that, we have to slightly re-define the scaling algebras $\underline{\mathfrak{F}}(O)$ when the underlying QFTGA satisfies Lorentz-covariance. For the remaining part of this article we adopt the following

Convention. Suppose that the underlying QFTGA satisfies also the condition of Lorentz-covariance (QFTGA.6). In this case, the local scaling algebras $\underline{\mathfrak{F}}(O)$, $O \in \mathcal{K}$, are defined as in Def. 2.2 but demanding in addition that the elements $\underline{F} \in \underline{\mathfrak{F}}(O)$ fulfill the also the condition

(e) $\quad || \, \underline{\tilde{\alpha}}_L(\underline{F}) - \underline{F} \, || \to 0$ as $L \to 1_{\tilde{\mathcal{L}}^\uparrow_+}$

where

$$(\underline{\tilde{\alpha}}_L(\underline{F}))_\lambda := \tilde{\mathscr{U}}(L)\underline{F}_\lambda \tilde{\mathscr{U}}(L)^*.$$

Again, it is not difficult to demonstrate that, with that convention, the $\underline{\mathfrak{F}}(O)$ are C^*-algebras containing plenty of elements, and $\underline{\alpha}_{\mathbb{R}^n}$, $\underline{\beta}_G$ and $\underline{\tilde{\alpha}}_{\tilde{\mathcal{L}}^\uparrow_+}$ act as strongly continuous groups of automorphisms on $\underline{\mathfrak{F}}$ with the covariance properties (2.4) and, in addition,

$$\underline{\tilde{\alpha}}_L(\underline{\mathfrak{F}}(O)) = \underline{\mathfrak{F}}(\Lambda(L)O), \quad L \in \tilde{\mathcal{L}}^\uparrow_+, \; O \in \mathcal{K}.$$

The following statement is again essentially a transcription of analogous results established for observable algebras in [5].

Proposition 3.1 *Suppose that the underlying QFTGA fulfills the conditions of Def. 2.2.*

1. *If the underlying QFTGA fulfills also Lorentz-covariance (QFTGA.6), then this property holds also for all scaling limit QFTGAs.*

2. *If the underlying QFTGA fulfills (QFTGA.6 & 7) and $n \geq 3$, then all scaling limit QFTGAs fulfill (QFTGA.6 & 7).*

3. *If the underlying QFTGA fulfills (QFTGA.8) and $n \geq 3$, then all scaling limit QFTGAs fulfill (QFTGA.7).*

4. *If the underlying QFTGA fulfills (QFTGA.6 & 9), then all scaling limit QFTGAs fulfill (QFTGA.6 & 9), too.*

Proof. Ad 1. This statement is proved in complete analogy to the corresponding statement in [5]; we note that for any scaling limit state $\omega_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega)$ (where ω is any locally normal state on $\mathfrak{F}$) there holds $\omega_{0,\iota} \circ \underline{\tilde{\alpha}}_L = \omega_{0,\iota}$ and hence one obtains a unitary representation of $\tilde{\mathfrak{L}}_+^{\uparrow}$ on $\mathcal{H}_{0,\iota}$ via setting

$$\tilde{\mathscr{U}}_{0,\iota}(L)\pi_{0,\iota}(\underline{F})\Omega_{0,\iota} := \pi_{0,\iota}(\underline{\tilde{\alpha}}_L(\underline{F}))\Omega_{0,\iota}\,, \quad L \in \tilde{\mathfrak{L}}_+^{\uparrow}\,,\ \underline{F} \in \underline{\mathfrak{F}}\,.$$

It is also easily checked that this unitary representation has all the properties analogous to those listed in (QFTGA.6) with respect to the scaling limit theory.

Ad 2. If the underlying theory has the additional properties (QFTGA.6 & 7), then this entails that the underlying theory also has the property (QFTGA.8) according to a result by Araki, Hepp and Ruelle [1]; cf. also the proof of Lemma 4.3 in [5]. The statement then follows from 1. and 3.

Ad 3. Let $\underline{F}'^{(1)}, \underline{F}'^{(2)} \in \underline{\mathfrak{F}}(O_{r'})$ and define, for some $h \in C_0^\infty(\mathbb{R}^n)$,

$$\underline{F}^{(j)} := \int d^n a\, h(a)\underline{\alpha}_a(\underline{F}'^{(j)})\,, \quad j = 1,2\,.$$

Then there is some $r > r'$ so that $\underline{F}^{(j)} \in \underline{\mathfrak{F}}(O_r)$, and clearly $\underline{F}_\lambda^{(j)} \in \mathcal{F}(\lambda O) \cap D(\partial_0)$. We apply the uniform clustering bound to obtain, for each $\lambda > 0$ and $|\boldsymbol{x}| > 3r$,

$$\begin{aligned}
&|\underline{\omega}_\lambda(\underline{F}^{(1)}\underline{\alpha}_{\boldsymbol{x}}(\underline{F}^{(2)})) - \underline{\omega}_\lambda(\underline{F}^{(1)})\underline{\omega}_\lambda(\underline{F}^{(2)})| \\
&= \ |\omega(\underline{F}_\lambda^{(1)}\alpha_{\lambda\boldsymbol{x}}(\underline{F}_\lambda^{(2)})) - \omega^\Omega(\underline{F}_\lambda^{(1)})\omega(\underline{F}_\lambda^{(2)})| \\
&\leq \ \frac{c(\lambda r)^{n-1}}{|\lambda\boldsymbol{x}|^{n-2}}(||\underline{F}_\lambda^{(1)}||\,||\partial_0(\underline{F}_\lambda^{(2)})|| + ||\partial_0(\underline{F}_\lambda^{(1)})||\,||\underline{F}_\lambda^{(2)}||) \\
&\leq \ \frac{cr^{n-1}}{|\boldsymbol{x}|^{n-2}}(||\,\underline{F}^{(1)}\,||\,||\,\underline{\partial}_0(\underline{F}^{(2)})\,|| + ||\,\underline{\partial}_0(\underline{F}^{(1)})\,||\,||\,\underline{F}^{(2)}\,||)\,,
\end{aligned}$$

where we have defined $\underline{\partial}_0(\underline{F}^{(j)}) := -i\,\frac{d}{dx^0}\big|_{x^0=0}\,\underline{\alpha}_{(x^0,\mathbf{0})}(\underline{F}^{(j)})$ and used the fact that $||\partial_0(\underline{F}_\lambda^{(j)})|| \leq \lambda^{-1}||\,\underline{\partial}_0(\underline{F}^{(j)})\,||$. Now $||\,\underline{\partial}_0(\underline{F}^{(j)})\,|| < \infty$ by the definition of the $\underline{F}^{(j)}$, and taking the $\limsup_\lambda$ on the left-hand side of the last inequality, one concludes that asymptotic spacelike clustering holds on the vacuum of each scaling limit theory since $\underline{F}^{(j)}$ approaches $\underline{F}'^{(j)}$ in the scaling algebra norm for $h \to \delta$. Because of normal commutation relations in each scaling limit QFTGA, this entails that $\mathfrak{F}_{0,\iota}' = \mathbb{C}\cdot 1$ holds in all scaling limit theories.

The proof proceeds analogously to the proof of Lemma 4.3 in [5]. □

There is another result worth mentioning here which also generalizes a corresponding result established for observable algebras in [5] and connects a duality condition in scaling limit theories with the type of the local von Neumann algebras of the underlying QFTGA.

Theorem 3.2 *Suppose that the underlying QFTGA fulfills the assumptions of Def. 2.1. Moreover, suppose that there exists a scaling limit QFTGA*

$$(\mathcal{F}_{0,\iota}, \mathscr{U}_{0,\iota}(\mathbb{R}^n), U_{0,\iota}^\bullet(G_{0,\iota}^\bullet), \Omega_{0,\iota}, k_{0,\iota}^\bullet)$$

having the property of "twisted wedge duality",

$$\mathcal{F}_{0,\iota}(W)' = \mathcal{F}^t_{0,\iota}(r_W(W))$$

for some wedge region W in $\mathbb{R}^d$ (with the definition of the twisted local von Newmann algebras analogous to (3.1) *and* (3.2) *with respect to the corresponding objects in the scaling limit QFTGA); moreover, suppose that $\mathfrak{F}_{0,\iota} \neq \mathbb{C} \cdot 1$. In this case it holds that the local von Neumann algebras $\mathcal{F}(O)$ are of type* III_1 *for each double cone $O \subset W$ whose boundary intersects $\overline{W} \cap \overline{r_W(W)}$. (The roles of W and $r_W(W)$ may be interchanged in this statement). If twisted wedge duality holds for all wedge regions in some scaling limit QFTGA, then one concludes that $\mathcal{F}(O)$ is of type* III_1 *for all scaling limit theories.*

We refer to Prop. 6.4 in [5] for a proof of this statement. We note also that according to the previous Proposition, the validity of conditions (QFTGA.6 & 7 & 9) in the underlying theory implies that the assumptions of Thm. 3.2 are fulfilled.

4 Quantum Field Systems with Gauge Symmetry

We now wish to investigate the scaling limits of QFTGAs that really correspond to superselection charges of a system of observables. Such QFTGAs are, more specifically, quantum field systems with gauge symmetry in the terminology of Doplicher and Roberts [8]. In order to summarize their definition here, and also for later reference, we first recapitulate some concepts of the Doplicher-Haag-Roberts approach to superselection theory, mainly from the sources [11, 20, 8].

This approach starts from the assumption that one is given an observable quantum system in a vacuum representation together with a further, distinguished set of representations modelling localized charges. The structure of an observable quantum system in a vacuum representation is described in terms of a collection of objects $(\mathcal{A}_{\mathrm{vac}}, \mathscr{U}_{\mathrm{vac}}(\mathbb{R}^n), \Omega_{\mathrm{vac}})$ whose properties are assumed to be as follows.

(a) $\mathcal{A}_{\mathrm{vac}}$ symbolizes a family $\{\mathcal{A}_{\mathrm{vac}}(O)\}_{O \in \mathcal{K}}$ of von Neumann algebras in a separable Hilbert space $\mathcal{H}_{\mathrm{vac}}$, subject to conditions of isotony (see above) and duality,

$$\mathcal{A}_{\mathrm{vac}}(O)' = \mathcal{A}_{\mathrm{vac}}(O') := \{\mathcal{A}_{\mathrm{vac}}(O_1) : \overline{O_1} \subset O' \,,\ O_1 \in \mathcal{K}\}'' \,,$$

where O' denotes the open causal complement of O. Setting moreover $\mathfrak{A}_{\mathrm{vac}} := \overline{\bigcup_O \mathcal{A}_{\mathrm{vac}}(O)}^{C^*}$, it is assumed that $\mathfrak{A}'_{\mathrm{vac}} = \mathbb{C} \cdot 1$.

(b) $\mathscr{U}_{\mathrm{vac}}(a)$, $a \in \mathbb{R}^n$, is a strongly continuous unitary representation of the translation group on $\mathcal{H}_{\mathrm{vac}}$, acting covariantly on the family $\{\mathcal{A}_{\mathrm{vac}}(O)\}_{O \in \mathcal{K}}$, and fulfilling the spectrum condition (see above). Furthermore, $\Omega_{\mathrm{vac}} \in \mathcal{H}_{\mathrm{vac}}$ is a unit vector which is let invariant by the action of $\mathscr{U}_{\mathrm{vac}}(a)$, $a \in \mathbb{R}^n$.

Remark. Usually, also the assumption is made that the family $\{\mathcal{A}_{\mathrm{vac}}(O)\}_{O \in \mathcal{K}}$ has the Borchers property ("Property B"). This property says that given $O, O_1 \in \mathcal{K}$ with $\overline{O} \subset O_1$ and a non-zero projection $E \in \mathcal{A}(O)$, then there is $V \in \mathcal{A}(O_1)$ with $VV^* = E$ and $V^*V = 1$. However, Roberts has shown [21] that this property can already be deduced from the other assumptions (essential being separability of $\mathcal{H}$ and the spectrum condition).

Given an observable quantum system $(\mathcal{A}_{\text{vac}}, \mathscr{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$, one may look for representations of $\mathfrak{A}_{\text{vac}}$ describing the presence of charges. Following Doplicher, Haag and Roberts, one may consider the set $\mathfrak{P}$ of representations π of $\mathfrak{A}_{\text{vac}}$ which are normal to the vacuum representation in restriction to the causal complement of any double cone. That means, if $\mathfrak{A}_{\text{vac}}(O')$ is defined as the C^*-algebra generated by all $\mathcal{A}_{\text{vac}}(O_1)$ where $\overline{O_1} \subset O'$, then π is in $\mathfrak{P}$ if $\pi \restriction \mathfrak{A}_{\text{vac}}(O')$ is normal to the identical representation of $\mathfrak{A}_{\text{vac}}(O')$ on $B(\mathcal{H}_{\text{vac}})$ for each $O \in \mathcal{K}$. Such representations describe superselection charges which are strictly localizable, see [11, 20] for further discussion. We shall be interested only in the subset $\mathfrak{P}_{\text{cov}}$ of those π in $\mathfrak{P}$ which are translation-covariant, meaning that there is a strongly continuous representation $\mathscr{U}_\pi(a)$, $a \in \mathbb{R}^n$, of the translation group on the representation-Hilbertspace of π fulfilling the spectrum condition and the intertwining property

$$\mathrm{Ad}\,\mathscr{U}_\pi(a)(\pi(A)) = \mathrm{Ad}\,\mathscr{U}_{\text{vac}}(a)(A)\,, \quad a \in \mathbb{R}^n,\ A \in \mathfrak{A}_{\text{vac}}\,.$$

By identifying the representation-Hilbertspace $\mathcal{H}_\pi$ with $\mathcal{H}_{\text{vac}}$, the set $\mathfrak{P}_{\text{cov}}$ may alternatively (and equivalently) be described in terms of the set Δ_t^{cov} of covariant, localized and transportable endomorphisms of $\mathfrak{A}_{\text{vac}}$. Here, an endomorphism $\rho : \mathfrak{A}_{\text{vac}} \to \mathfrak{A}_{\text{vac}}$ is called localized in $O \in \mathcal{K}$ if $\rho(A) = A$ holds for all $A \in \mathfrak{A}_{\text{vac}}(O')$. It is called transportable if, given an arbitrary region $O_1 \in \mathcal{K}$, there exists a unitary V so that $V\rho(\,.\,)V^*$ is an endomorphism of $\mathfrak{A}$ localized in O_1; one can show that V may be chosen as an element of $\mathfrak{A}$.

An element $\rho \in \Delta_t^{\text{cov}}$ is called *irreducible* if $\rho(\mathfrak{A}_{\text{vac}})' = \mathbb{C}\cdot 1$, and the set Sect^{cov} of all equivalence classes

$$[\rho] := \{V\rho(\,.\,)V^* : V^* = V^{-1} \in \mathfrak{A}_{\text{vac}}\}$$

for irreducible $\rho \in \Delta_t^{\text{cov}}$ is called the set of *translation-covariant superselection sectors* of the given observable quantum system $(\mathcal{A}_{\text{vac}}, \mathscr{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$.

If $\rho, \rho' \in \Delta_t^{\text{cov}}$, one defines by $\mathcal{I}(\rho, \rho')$ the set of *intertwiners* between ρ and ρ' as the set of all $T \in \mathfrak{A}_{\text{vac}}$ which satisfy

$$T\rho(A) = \rho'(A)T\,, \quad A \in \mathfrak{A}_{\text{vac}}\,.$$

Strictly speaking, one should refer to $\mathcal{I}(\rho, \rho')$ as the set of **global intertwiners** between ρ and ρ'. Given $O_1 \in \mathcal{K}$ and $\rho, \rho' \in \Delta_t^{\text{cov}}$ localized in O_1, one can introduce $\mathcal{I}(\rho, \rho')_O$, the set of **local intertwiners** with respect to the localization region $O \supset O_1$, as consisting of all $T \in \mathfrak{A}_{\text{vac}}$ fulfilling

$$T\rho(A) = \rho'(A)T\,, \quad A \in \mathcal{A}_{\text{vac}}(O)\,.$$

Hence it is obvious that $\mathcal{I}(\rho, \rho')_O \supset \mathcal{I}(\rho, \rho')$ for all $O \in \mathcal{K}$, and in Sec. 6 we will link the question if local and global intertwiners are equivalent, i.e. if $\mathcal{I}(\rho, \rho')_O = \mathcal{I}(\rho, \rho')$ holds for all $O \in \mathcal{K}$, to the preservance of charges in the scaling limit.

Presently, we need to very briefly summarize some further concepts of charge superselection theory (see, e.g. [20] for a more detailed account). First, one can introduce for $T_1 \in \mathcal{I}(\rho_1, \rho_1')$ and $T_2 \in \mathcal{I}(\rho_2, \rho_2')$ a product operation $T_1 \times T_2$ yielding an element in $\mathcal{I}(\rho_1\rho_2, \rho_1'\rho_2')$. There is then a distinguished familiy of intertwiners $\epsilon(\rho_1, \rho_2) \in \mathcal{I}(\rho_1\rho_2, \rho_2\rho_1)$, for irreducible $\rho_1, \rho_2 \in \Delta_t^{\mathrm{cov}}$, characterized by the property that it describes the exchange in the intertwiner product according to

$$(T_2 \times T_2)\epsilon(\rho_1, \rho_2) = \epsilon(\rho_1', \rho_2')(T_1 \times T_2)\,, \quad T_j \in \mathcal{I}(\rho_j, \rho_j')\,,$$

together with the properties $\epsilon(\rho_1, \rho_2) = 1_{\rho_1\rho_2}$ if the localization regions of ρ_1 and ρ_2 are spacelike separated, and $\epsilon(\rho_2, \rho_1)\epsilon(\rho_1, \rho_2) = 1_{\rho_1\rho_2}$. Moreover, one can show that each irreducible $\rho \in \Delta_t^{\mathrm{cov}}$ posseses a left inverse φ_ρ, i.e. a positive linear map on $\mathfrak{A}_{\mathrm{vac}}$ which preserves the unit and fulfills $\varphi_\rho(A\rho(B)) = \varphi_\rho(A)B$. Then there is for ρ a number λ_ρ so that

$$\varphi_\rho(\epsilon(\rho, \rho)) = \lambda_\rho 1.$$

The number λ_ρ depends only on the equivalence class $[\rho]$ of ρ and is called the *statistics parameter* of the corresponding superselection sector. If $\lambda_\rho \neq 0$, then the superselection sector is said to have *finite statistics*. We define by $\mathrm{Sect}_{\mathrm{fin}}^{\mathrm{cov}}$ the set of all translation-covariant superselection sectors of the underlying observable quantum system which have finite statistics, and by $\Delta_{\mathrm{fin}}^{\mathrm{cov}}$ the set of all endomorphisms ρ with $[\rho] \in \mathrm{Sect}_{\mathrm{fin}}^{\mathrm{cov}}$.

Finally, we need to recollect the notion of a conjugate charge. One can show (cf. e.g. [20]) that for each $\rho \in \Delta_{\mathrm{fin}}^{\mathrm{cov}}$ localized in $O \in \mathcal{K}$ there is some $\overline{\rho} \in \Delta_{\mathrm{fin}}^{\mathrm{cov}}$, also localized in O, together with isometries R and $\overline{R}$ in $\mathcal{A}(O)$ which intertwine the endomorphisms $\overline{\rho}\rho$ and $\rho\overline{\rho}$, respectively, with the identical endomorphism of $\mathfrak{A}_{\mathrm{vac}}$, that is,

$$\overline{\rho}(\rho(A))R = RA \quad \text{and} \quad \rho(\overline{\rho}(A))\overline{R} = \overline{R}A\,, \quad A \in \mathfrak{A}_{\mathrm{vac}}\,.$$

In this case, one calles $[\overline{\rho}]$ the conjugate superselection sector of $[\rho]$ or, synonymously, the conjugate charge of $[\rho]$.

Doplicher and Roberts [8] have shown that one can construct from $\Delta_{\mathrm{fin}}^{\mathrm{cov}}$ and the interwiners a system of local field algebras, acted upon by a faithful unitary representation of a compact group — called the gauge group — such that the local algebras of the initially given observable quantum system are embedded in the local field algebras as exactly containing the invariant elements under the gauge group action. In more precise terms, they have shown that one can associate with $(\mathcal{A}_{\mathrm{vac}}, \mathscr{U}_{\mathrm{vac}}, \Omega_{\mathrm{vac}})$ a *quantum field system with gauge symmetry* (QFSGS), defined as follows:

Definition 4.1 $(\mathcal{F}, \mathscr{U}(\mathbb{R}^n), U(G), \Omega, k)$ is a QFSGS for $(\mathcal{A}_{\mathrm{vac}}, \mathscr{U}_{\mathrm{vac}}, \Omega_{\mathrm{vac}})$ and $\Delta_{\mathrm{fin}}^{\mathrm{cov}}$ if the following conditions hold:

(QFSGS.1) $(\mathcal{F}, \mathscr{U}(\mathbb{R}^n), U(G), \Omega, k)$ is a QFTGA; the Hilbert space on which the von Neumann algebras $\mathcal{F}(O)$ of $\mathcal{F} = \{\mathcal{F}(O)\}_{O \in \mathcal{K}}$ act will be denoted by $\mathcal{H}$. Moreover, $\mathfrak{F}' = \mathbb{C}\mathbf{1}$.

(QFSGS.2) There is a C^*-algebraic monomorphism

$$\pi : \mathfrak{A}_{\mathrm{vac}} \to \mathfrak{F}$$

so that $\pi(\mathcal{A}_{\rm vac}(O))$ consists exactly of all $A \in \mathcal{F}(O)$ having the property that $U(g)AU(g)^* = A$ holds for all $g \in G$. We will use the shorter notation

$$\mathcal{A}(O) := \pi(\mathcal{A}_{\rm vac}(O)) .$$

(QFSGS.3) Let $[\rho] \in \mathrm{Sect}^{\rm cov}_{\rm fin}$ be a superselection sector. Then there exists a finite dimensional, irreducible, unitary representation

$$v_{[\rho]} = (v_{[\rho]ji})^d_{i,j=1}$$

of G (acting as a matrix representation for some suitable $d = d_{[\rho]}$) so that, for each $O \in \mathcal{K}$, there is a multiplet $\psi_1, \dots, \psi_d$ of elements in $\mathcal{F}(O)$ having the following properties:

$$U(g)\psi_i U(g)^* = \sum_{j=1}^{d} \psi_j v_{[\rho]ji}(g) \,, \tag{4.1}$$

$$\psi_i^*\psi_j = \delta_{ij}\mathbf{1} \,, \qquad \sum_{j=1}^{d} \psi_j\psi_j^* = \mathbf{1} \,, \tag{4.2}$$

$$\pi \circ \rho_O(A) = \sum_{j=1}^{d} \psi_j \pi(A)\psi_j^* \,, \quad A \in \mathfrak{A}_{\rm vac} \,, \tag{4.3}$$

for some representer ρ_O of $[\rho]$ localized in O.

These properties fix $v_{[\rho]}$ to within unitary equivalence.

(QFSGS.4) $\mathcal{F}(O)$ is generated by $\mathcal{A}(O)$ and all multiplets ψ_j, $j = 1, \dots, d_{[\rho]}$, with the properties (4.1),(4.2), (4.3), as $[\rho]$ ranges over all superselection sectors in $\mathrm{Sect}^{\rm cov}_{\rm fin}$. For each finite-diemsional, irreducible, unitary representation v of G there is some superselection sector $[\rho] \in \mathrm{Sect}^{\rm cov}_{\rm fin}$ so that $v = v_{[\rho]}$ where $v_{[\rho]}$ has the properties of (QFSGS.3).

The conditions for a QFSGS associated with $(\mathcal{A}_{\rm vac}, \mathscr{U}_{\rm vac}(\mathbb{R}^n), \Omega_{\rm vac})$ and $\Delta^{\rm cov}_{\rm fin}$ are given here in a form slightly different from the statement in [8]; however, the present formulation is convenient for our purposes.

It is plain that a QFSGS is a QFTGA fulfilling additional properties. Condition (QFSGS.4) states, in particular, that $\mathrm{Sect}^{\rm cov}_{\rm fin}$ can be identified with the dual group, $\widehat{G}$, of the gauge group G. The connection between field algebra and superselection sectors is essentially expressed through the multiplet operators $\psi_1, \dots, \psi_d$ with the properties listed in (QFSGS.3). In fact, the occurrence of such "charge multiplets" associated with the superselection sector $[\rho]$ is equivalent to the presence of the corresponding charge in the QFSGS $(\mathcal{F}, \mathscr{U}(\mathbb{R}^n), U(G), \Omega, k)$. This will, basically, be our starting point for formulating criteria that express "preservation of a charge" in the scaling limit.

5 Preservance of Charges in the Scaling Limit

Let us now discuss the problem of characterizing "preservation of charges in the scaling limit" in greater detail. To this end, let $(\mathcal{F}, \mathscr{U}(\mathbb{R}^n), U(G), \Omega, k)$ be a QFSGS associated with $(\mathcal{A}_{\text{vac}}, \mathscr{U}_{\text{vac}}(\mathbb{R}^n), \Omega_{\text{vac}})$ and $\Delta_{\text{fin}}^{\text{cov}}$. Since $(\mathcal{F}, \mathscr{U}(\mathbb{R}^n), U(G), \Omega, k)$ is a QFTGA, we can form the corresponding scaling algebra $\underline{\mathfrak{F}}$ as in Sec. 2. We may then define

$$\underline{\mathfrak{A}}(O) = \{\underline{A} \in \underline{\mathfrak{F}}(O) : \underline{A}_\lambda \in \mathcal{A}(\lambda O)\},$$

and it is not difficult to see that $\underline{\mathfrak{A}}(O)$ consists exactly of the $\underline{A} \in \underline{\mathfrak{F}}(O)$ so that

$$\underline{\beta}_g(\underline{A}) = \underline{A}$$

for all $g \in G$.

Now let $\omega_{0,\iota} \in \mathrm{SL}^{\underline{\mathfrak{F}}}(\omega)$ be a scaling limit state on $\underline{\mathfrak{F}}$, and denote by

$$(\mathfrak{F}_{0,\iota}, \mathscr{U}_{0,\iota}(\mathbb{R}^n), U^\bullet_{0,\iota}(G^\bullet_{0,\iota}), \Omega_{0,\iota}, k^\bullet_{0,\iota})$$

the corresponding scaling limit QFTGA. Let us also denote by

$$\mathcal{A}_{0,\iota}(O) = \pi_{0,\iota}(\underline{\mathfrak{A}}(O))'', \quad O \in \mathcal{K},$$

the von Neumann algebra formed by the scaling limits of the observables of the underlying QFSGS, and define by

$$\mathcal{F}_{0,\iota}(O)^{G^\bullet_{0,\iota}} = \{\boldsymbol{f} \in \mathcal{F}_{0,\iota}(O) : U^\bullet_{0,\iota}(g^\bullet)\boldsymbol{f} = \boldsymbol{f}U^\bullet_{0,\iota}(g^\bullet) \ \forall\, g^\bullet \in G^\bullet_{0,\iota}\}$$

the fixed point algebra of the gauge group action in the scaling limit. With this notation, and recalling that $\mathcal{H}_{0,\iota} = \overline{\mathfrak{F}_{0,\iota}\Omega_{0,\iota}}$, we find:

Lemma 5.1 *(i)* $\mathcal{A}_{0,\iota}(O) = \mathcal{F}_{0,\iota}(O)^{G^\bullet_{0,\iota}}$, $O \in \mathcal{K}$.
(ii) Suppose that $\Omega_{0,\iota}$ *is the unique (up to a phase) unit vector in* $\mathcal{H}_{0,\iota}$ *which is invariant under* $\mathscr{U}_{0,\iota}(\mathbb{R}^n)$ *(equivalently,* $\mathfrak{F}'_{0,\iota} = \mathbb{C}\cdot 1$*). If* $\mathfrak{A}_{0,\iota} = \overline{\bigcup_O \mathcal{A}_{0,\iota}(O)}^{C^*}$ *is abelian, then* $\mathfrak{F}_{0,\iota} = \mathbb{C}\cdot 1$ *and hence,* $\mathcal{H}_{0,\iota} = \mathbb{C}\Omega_{0,\iota}$.

Proof. (i) Clearly, one has $\mathcal{A}_{0,\iota}(O) \subset \mathcal{F}_{0,\iota}(O)^{G^\bullet_{0,\iota}}$. To show that the reverse inclusion holds, let $\boldsymbol{f} \in \mathcal{F}_{0,\iota}(O)$. Denote by $m_{0,\iota}(\boldsymbol{h}) = \int_G d\mu(g)\, U_{0,\iota}(g)\boldsymbol{h}U_{0,\iota}(g)^*$, $\boldsymbol{h} \in \mathfrak{F}_{0,\iota}$, the mean over the action of G on $\mathfrak{F}_{0,\iota}$. We have $m_{0,\iota}(\boldsymbol{f}) = \boldsymbol{f}$. Let $\underline{F}^{(n)}$, $n \in \mathbb{N}$, be a sequence of elements in $\underline{\mathfrak{F}}(O)$ so that $w\text{-}\lim_{n\to\infty} \pi_{0,\iota}(\underline{F}^{(n)}) = \boldsymbol{f}$. Such a sequence exists because, by a Reeh-Schlieder argument, $\Omega_{0,\iota}$ is separating for $\mathcal{F}_{0,\iota}(O)$. Using this separating property of $\Omega_{0,\iota}$ once more, also $m_{0,\iota}(\pi_{0,\iota}(\underline{F}^{(n)}))$ approximates $\boldsymbol{f}$ weakly. On the other hand,

$$m_{0,\iota}(\pi_{0,\iota}(\underline{F}^{(n)})) = \int_G d\mu(g)\, \pi_{0,\iota}(\underline{\beta}_g(\underline{F}^{(n)})) = \pi_{0,\iota}(\int_G d\mu(g)\, \underline{\beta}_g(\underline{F}^{(n)})),$$

where we made use of the continuity of $\underline{\beta}_G$ in norm on the scaling algebra to interchange representation and integration. Since $\int_G d\mu(g)\, \underline{\beta}_g(\underline{F}^{(n)})$ is contained in $\underline{\mathfrak{A}}(O)$, we see that $\boldsymbol{f}$ is weakly approximated by elements in $\mathcal{A}_{0,\iota}(O)$ and hence is itself contained in $\mathcal{A}_{0,\iota}(O)$.

(ii) Under the given hypotheses, a result by Buchholz (Lemma 3.1 in [3]) shows that $\mathfrak{A}_{0,\iota} = \mathbb{C} \cdot 1$. Hence, the strongly continuous group $\beta_g^{(0,\iota)} = \mathrm{Ad}\, U_{0,\iota}(g)$, $g \in G$, of automorphisms on $\mathfrak{F}_{0,\iota}$ acts ergodically, meaning that $\beta_g^{(0,\iota)}(\boldsymbol{f}) = \boldsymbol{f}$ for all $g \in G$ implies $\boldsymbol{f} \in \mathbb{C} \cdot 1$. Using Thm. 4.1 in [13], it follows that the unique ergodic state for $\beta_G^{(0,\iota)}$ on $\mathfrak{F}_{0,\iota}$ is a trace. The scaling limit vacuum $\langle \Omega_{0,\iota}, \,.\, \Omega_{0,\iota} \rangle$ is a pure $\beta_G^{(0,\iota)}$-invariant state on $\mathfrak{F}_{0,\iota}$ and hence is a trace. (Purity of this state holds since the space of translation-invariant vectors in $\mathcal{H}_{0,\iota}$ is one-dimensional.) This implies

$$\langle \Omega_{0,\iota}, \boldsymbol{f}^* \mathscr{U}_{0,\iota}(x) \boldsymbol{f} \Omega_{0,\iota} \rangle = \langle \Omega_{0,\iota}, \boldsymbol{f} \mathscr{U}_{0,\iota}(-x) \boldsymbol{f}^* \Omega_{0,\iota} \rangle$$

for each $\boldsymbol{f} \in \mathcal{F}_{0,\iota}(O)$, $O \in \mathcal{K}$, and all $x \in \mathbb{R}^n$. Arguing with spectrum condition and clustering (as a consequence of the assumption that every translation-invariant vector in $\mathcal{H}_{0,\iota}$ is a multiple of $\Omega_{0,\iota}$) in the same manner as in the proof of Lemma 3.1 in [3], one concludes that $\boldsymbol{f} \in \mathbb{C} \cdot 1$. Hence $\mathfrak{F}_{0,\iota} = \mathbb{C} \cdot 1$. □

The Lemma shows that all charges of the underlying QFSGS disappear in a scaling limit theory once the scaling limit theory is known to be classical for the observables, provided the underlying theory satisfies very general conditions such as clustering (QFTGA.8) or (for $n \geq 3$) Lorentz-covariance (QFTGA.6).

At this point, we should emphasize the distinction between charges in the scaling limit QFTGA which are "scaling limits of charges of the underlying QFSGS", and "charges arising as superselection sectors of the scaling limit theory", as was first discussed by D. Buchholz [2]. Charges of the first mentioned type correspond to the situation that $G_{0,\iota}^{\bullet}$ is non-trivial and hence $U_{0,\iota}^{\bullet}(G_{0,\iota}^{\bullet})$ acts non-trivially (and faithfully) on $\mathfrak{F}_{0,\iota}$. In this case, the action of $U_{0,\iota}^{\bullet}(G_{0,\iota}^{\bullet})$ on the elements of $\mathfrak{F}_{0,\iota}$ may be seen as a short-distance remnant of the action of $U(G)$ on $\mathfrak{F}$ so that, correspondingly, the members of the spectrum $\widehat{G}_{0,\iota}^{\bullet}$ of $G_{0,\iota}^{\bullet}$ may be viewed as representing short-distance remnants of the charges in $\widehat{G}$ of the underlying QFSGS. It is important to note that, to some extent, these charges of the scaling limit theory have been present in the underlying QFSGS. We will discuss this case in more detail below.

The second type of charges in the scaling limit arises in a different way. One may consider the scaling limit theory (induced by $\omega_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega)$)

$$(\mathcal{A}_{0,\iota}, \mathscr{U}_{0,\iota}(\mathbb{R}^n), \Omega_{0,\iota})$$

which is gained form the observables of the underlying QFSGS as a new observable quantum system in its own right (provided it fulfills the assumptions of irreducibility). Then one can assign a set of superselection sectors $\mathrm{Sect}_{\mathrm{fin}}^{\mathrm{cov}} = \mathrm{Sect}_{\mathrm{fin}}^{\mathrm{cov}}(\mathcal{A}_{0,\iota})$ to this observable quantum system, and by the Doplicher-Roberts reconstruction theorem, we can now associate to these data a QFSGS, which we may denote by

$$(\mathcal{F}^{(0,\iota)}, \mathscr{U}^{(0,\iota)}(\mathbb{R}^n), U^{(0,\iota)}(G^{(0,\iota)}), \Omega^{(0,\iota)}, k^{(0,\iota)})\,.$$

Thus, this QFSGS contains the superselection charges which arise in the scaling limit theory of the *observables* of the underlying QFSGS. In general, it may occur that $\mathfrak{F}_{0,\iota}$ is properly contained in $\mathfrak{F}^{(0,\iota)}$ and that $G_{0,\iota}^{\bullet}$ is a factor group of $G^{(0,\iota)}$ by some non-trivial normal subgroup, so that the QFTGA associated with $\mathfrak{F}_{0,\iota}$ may be viewed as a proper subtheory (in the sense of [8]) of the QFSGS associated with $\mathfrak{F}^{(0,\iota)}$. Buchholz [2] proposed to consider such a case as a criterion for confinement, since it models the situation where charges appear as superselection sectors of the (observables') scaling limit theory which do not arise as scaling limits of charges that occur as superselection sectors in the underlying QFSGS. We refer to [2, 4] for further discussion, and we note that examples for superselection charges of this second type have been constructed for the Schwinger model in two spacetime dimensions [4, 6].

In the present work, we shall restrict attention solely to charges in the scaling limit QFTGAs of an underlying QFSGS of the first mentioned type, i.e. which arise as "scaling limits" of charges present in the underlying QFSGS. Having clarified this basic point, we must find criteria which express that a charge of the underlying QFSGS has a non-trivial scaling limit. There are some prefatory observations which may be helpful as a guideline. We have already seen that the gauge group $G^\bullet_{0,\iota} = G_{0,\iota}/N_{0,\iota}$ of a scaling limit QFTGA is a factor group of $G_{0,\iota}$ which is itself a copy of G, the gauge group of the underlying QFSGS. It may in general happen that the normal subgroup $N_{0,\iota}$ is non-trivial, and hence that $G^\bullet_{0,\iota}$ is "smaller" than G. In this situation, certainly not all the charges of the underlying QFSGS will have counterparts in the scaling limit QFTGA. Thus, we will in general be confronted with a situation which is in a sense complentary to that of $\mathfrak{F}_{0,\iota} \subset \mathfrak{F}^{(0,\iota)}$ mentioned just before and where, morally, the scaling limit QFTGA associated with $\mathfrak{F}_{0,\iota}$ corresponds to a subtheory of the underlying QFSGS, at least as far as the charge structure is concerned.[4] However, since there is no inclusion of $\mathfrak{F}_{0,\iota}$ into $\mathfrak{F}$, we need to establish a correspondence between elements in $\mathfrak{F}_{0,\iota}$ and in $\mathfrak{F}$ which allows to decide if charges present in the underlying QFSGS are also present in the scaling limit.

As we have mentioned above, the presence of a superselection charge in the underlying QFSGS manifests itself through the presence of charge multiplets $\psi_1, \dots, \psi_d \in \mathfrak{F}$ which transform under a finite dimensional, irreducible, unitary representation $v_{[\rho]}$ as described in (QFSGS.3). This will be the starting point for our criterion of charge preservance in the scaling limit. To fix ideas, let $(\mathcal{F}, \mathscr{U}(\mathbb{R}^n), U(G), \Omega, k)$ denote the underlying QFSGS, and let $[\rho] \in \mathrm{Sect}^{\mathrm{cov}}_{\mathrm{fin}}$ be one of its superselection sectors, and pick some arbitrary $O \in \mathcal{K}$. Then there is a finite-dimensional, irreducible, unitary representation $v_{[\rho]}$ of G and, for each $\lambda > 0$, a multiplet of elements $\psi_1(\lambda), \dots, \psi_d(\lambda)$ in $\mathcal{F}(\lambda O)$ having the properties of (QFSGS.3) with respect to the localization region λO. We will refer to any such multiplet family $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ as a **scaled multiplet** for $[\rho]$. The principal idea is now to view the functions $\lambda \mapsto \psi_j(\lambda)$ as "would-be" elements of $\underline{\mathfrak{F}}(O)$ and to follow their fate as λ approches 0. However, these functions won't satisfy the "phase-space constraint" condition (c) of Def. 2.2 which is essential in order to interpret them as orbits of field algebra elements under (abstract) renormalization group transformations. Hence, if $\omega_{0,\iota}$ is a scaling limit state, in general one can't form $\pi_{0,\iota}(\psi_j(\,.\,))$ since $\psi_j(\,.\,)$ won't belong to the scaling algebra $\underline{\mathfrak{F}}$. But one can still check if, in the scaling limit, scaled multiplets become close to elements of $\pi_{0,\iota}(\underline{\mathfrak{F}})$ so that they can effectively be regarded as representing elements in the scaling limit von Neumann algebras $\mathcal{F}_{0,\iota}(O) = \pi_{0,\iota}(\underline{\mathfrak{F}}(O))''$. We will introduce some new terminology which gives this idea a more precise shape.

[4] The dynamics of the theories corresponding to $\mathfrak{F}_{0,\iota}$ and $\mathfrak{F}$ are expected to be different and so the former can't be a subtheory of the latter in the full sense of the definition.

Definition 5.2 Let $\omega_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega)$ be a scaling limit state of the underlying QFSGS. Then we say that a family $\{f(\lambda)\}_{\lambda>0}$ fulfilling (i) $f(\lambda) \in \mathfrak{F}(\lambda O_1)$ for some $O_1 \in \mathcal{K}$, (ii) $\sup_{\lambda>0} ||f(\lambda)|| < \infty$, and (iii) $\sup_\lambda ||\beta_g(f(\lambda)) - f(\lambda)|| \to 0$ for $g \to 1_G$, is *asymptotically contained* in $\mathcal{F}_{0,\iota}(O)$ if the following holds:

For each given $\epsilon > 0$ there are elements $\underline{F}$ and $\underline{F}'$ in $\underline{\mathfrak{F}}(O)$ such that

$$\limsup_\kappa \left(||(f(\lambda_\kappa) - \underline{F}_{\lambda_\kappa})\Omega|| + ||(f(\lambda_\kappa) - \underline{F}'_{\lambda_\kappa})^*\Omega||\right) < \epsilon\,, \tag{5.1}$$

where the net $\{\lambda_\kappa\}_{\kappa\in K}$ of positive numbers converges to 0, with $\omega_{0,\iota} = \lim_\kappa \underline{\omega}_{\lambda_\kappa}$ on $\underline{\mathfrak{F}}$.

Let us collect some immediate results related to this definition.

Lemma 5.3 *Let $\omega_{0,\iota}$ be a scaling limit state of the underlying QFSGS, and suppose that $\{f(\lambda)\}_{\lambda>0}$ is a family of elements in $\mathfrak{F}$ with the properties as in the previous definition. Then the following statements are equivalent:*

(*a*) *$\{f(\lambda)\}_{\lambda>0}$ is asymptotically contained in $\mathcal{F}_{0,\iota}(O)$ for all $O \supset \overline{O_1}$,*

(*b*) *In the scaling limit, $\{f(\lambda)\}_{\lambda>0}$ is approached in the $*$-strong topology by elements in $\pi_{0,\iota}(\underline{\mathfrak{F}}(O))$ in the following sense: Whenever $O \supset \overline{O_1}$, $\epsilon > 0$ and finitely many $\underline{F}^{(1)}, \dots, \underline{F}^{(N)} \in \underline{\mathfrak{F}}$ are given, then there is an $\underline{F} \in \underline{\mathfrak{F}}(O)$ fulfilling $||\underline{F}|| \le \sup_\lambda ||f(\lambda)||$ and*

$$\limsup_\kappa \left(||(f(\lambda_\kappa) - \underline{F}_{\lambda_\kappa})\underline{F}^{(j)}_{\lambda_\kappa}\Omega|| + ||(f(\lambda_\kappa) - \underline{F}_{\lambda_\kappa})^*\underline{F}^{(j)}_{\lambda_\kappa}\Omega||\right) < \epsilon\,, \quad j = 1, \dots, N\,,$$

where $\{\lambda_\kappa\}_{\kappa\in K}$ is as in the previous definition,

(*c*) *There is for each $\epsilon > 0$ and for each $h \in L^1(\mathbb{R}^n)$ having compact support, with $h \ge 0$ and $\int d^n x\, h(x) = 1$, some number $\mu_0 > 0$ so that*

$$\limsup_\kappa \left(||\,(\,(\underline{\alpha}_{h_\mu} f)(\lambda_\kappa) - f(\lambda_k)\,)\Omega\,|| + ||\,(\,(\underline{\alpha}_{h_\mu} f)(\lambda_\kappa) - f(\lambda_k)\,)^*\Omega\,||\right) < \epsilon\,, \tag{5.2}$$

for all $0 < \mu < \mu_0$, where $h_\mu(x) = \mu^{-n} h(x/\mu)$ and

$$(\underline{\alpha}_h f)(\lambda) = \int d^n x\, h(x)\alpha_{\lambda x}(f(\lambda))\,, \quad \lambda > 0 \;\; h \in L^1(\mathbb{R}^n)\,.$$

(The latter integral is to be interpreted in the weak topology on $\mathfrak{F}$; $\{\lambda_\kappa\}_{\kappa\in K}$ is as before.)

Proof. (*a*) $\Rightarrow$ (*c*). Writing $(\underline{\alpha}_h \underline{F})_\lambda = \int d^n x\, h(x)\alpha_{\lambda x}(\underline{F}_\lambda)$, we consider the estimate

$$\begin{aligned} ||(\,(\underline{\alpha}_{h_\mu} f)(\lambda_\kappa) - f(\lambda_\kappa)\,)\Omega|| \;\le\; & ||(\,(\underline{\alpha}_{h_\mu} f)(\lambda_\kappa) - (\underline{\alpha}_{h_\mu}\underline{F})_{\lambda_\kappa}\,)\Omega|| + ||(\,(\underline{\alpha}_{h_\mu}\underline{F})_{\lambda_\kappa} - \underline{F}_{\lambda_\kappa}\,)\Omega|| \\ & + ||(\underline{F}_{\lambda_\kappa} - f(\lambda_\kappa))\Omega||\,. \end{aligned}$$

Denoting by $\hat{h}$ the Fourier transform of h and by $P = (P_\nu)_{\nu=0}^{n-1}$ the selfadjoint generators of the unitary translation group of the underlying QFSGS, the first term on the right hand side is seen to equal

$$||\hat{h}(\mu P)(f(\lambda_\kappa) - \underline{F}_{\lambda_\kappa})\Omega|| \le ||h||_{L^1}||(f(\lambda_\kappa) - \underline{F}_{\lambda_\kappa})\Omega||\,.$$

The second term on the right hand side can be estimated by

$$||h||_{L^1} \sup_{x\in\operatorname{supp} h} ||\underline{\alpha}_{\mu x}(\underline{F}) - \underline{F}||$$

and tends to 0 as $\mu \to 0$ for $\underline{F} \in \underline{\mathfrak{F}}(O)$. Using these estimates, it is easy to see that (a) implies (c).

$(c) \Rightarrow (b)$. It holds that $\lambda \mapsto \underline{\Phi}_\lambda = (\underline{\alpha}_{h_\mu} f)(\lambda)$ is contained in $\underline{\mathfrak{F}}(O_\times)$ where $O_\times$ is any double cone containing $O_1 + \operatorname{supp} h_\mu$. A standard Reeh-Schlieder argument shows that, if W is any wedge region in the causal complement of $O_\times$, then $\mathcal{F}_{0,\iota}(W)\Omega_{0,\iota}$ is dense in $\mathcal{H}_{0,\iota}$. As a consequence, there is for given $\underline{F}^{(j)} \in \underline{\mathfrak{F}}$ and given $\eta > 0$ some $B^{(j)} \in \underline{\mathfrak{F}}(W)$ so that

$$||(\pi_{0,\iota}(\underline{F}^{(j)}) - \pi_{0,\iota}(\underline{B}^{(j)}))\Omega_{0,\iota}|| < \eta.$$

Thus, making first η and then μ small enough, one can arrange that

$$\begin{aligned}
&\limsup_\kappa \Big(||(\underline{\Phi}_{\lambda_\kappa} - f(\lambda_\kappa))\underline{F}^{(j)}_{\lambda_\kappa}\Omega|| + ||(\underline{\Phi}_{\lambda_\kappa} - f(\lambda_\kappa))^*\underline{F}^{(j)}_{\lambda_\kappa}\Omega||\Big) \\
&\quad \leq \limsup_\kappa \Big(||(\underline{\Phi}_{\lambda_\kappa} - f(\lambda_\kappa))\underline{B}^{(j)}_{\lambda_\kappa}\Omega|| + ||(\underline{\Phi}_{\lambda_\kappa} - f(\lambda_\kappa))^*\underline{B}^{(j)}_{\lambda_\kappa}\Omega||\Big) + 4\eta \sup_\kappa ||f(\lambda_\kappa)|| \\
&\quad = \limsup_\kappa \Big(||\underline{B}^{(j)}_{\lambda_\kappa}(\underline{\Phi}_{\lambda_\kappa} - f(\lambda_\kappa))\Omega|| + ||\underline{B}^{(j)}_{\lambda_\kappa}(\underline{\Phi}_{\lambda_\kappa} - f(\lambda_\kappa))^*\Omega||\Big) + 4\eta \sup_\kappa ||f(\lambda_\kappa)|| \\
&\quad \leq \limsup_\kappa ||\underline{B}^{(j)}|| \big(||(\underline{\Phi}_{\lambda_\kappa} - f(\lambda_\kappa))\Omega|| + ||(\underline{\Phi}_{\lambda_\kappa} - f(\lambda_\kappa))^*\Omega||\big) + 4\eta \sup_\kappa ||f(\lambda_\kappa)||
\end{aligned}$$

can be made smaller than any given $\epsilon > 0$; then, for a sufficiently small μ, $\underline{\Phi}$ can be taken as the $\underline{F}$ required in (b). Note that in passing from the second line to the third we have used that $V\underline{B}^{(j)}_\lambda V^*$ commutes with $V(\underline{\Phi}_\lambda - f(\lambda))V^*$ and its adjoint, where V is the unitary "twist" operator defined in (3.2), because of the localization properties of the operators involved; moreover, $V\Omega = \Omega$.

The implication $(b) \Rightarrow (a)$ is obvious. □

Remark. In view of statement (b) of the previous Lemma, one might refer to our notion of asymptotic containment more precisely as $*$-strong asymptotic containment. It should then be obvious how to introduce, e.g., the notion of strong or weak asymptotic containment in $\mathcal{F}_{0,\iota}(O)$ for families $\{f(\lambda)\}_{\lambda>0}$ fulfilling the properties as in 5.2. One could also drop condition (iii) on $\{f(\lambda)\}_{\lambda>0}$ in the definition of asymptotic containment, then having to define in Lemma 5.3 $\underline{\alpha}_h f$ differently, cf. (2.5).

After these preparations, we can now present our criterion for preservance of charges in the scaling limit.

Definition 5.4 Let $\omega_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega)$ be a scaling limit state of the underlying QFSGS, and let $[\rho] \in \mathrm{Sect}^{\mathrm{cov}}_{\mathrm{fin}}$ be a superselection sector. Then we say that the charge $[\rho]$ is *preserved in the scaling limit QFTGA of* $\omega_{0,\iota}$ if, for each $O_1 \in \mathcal{K}$, there is some scaled multiplet $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ for $[\rho]$ with $\psi_j(\lambda) \in \mathcal{F}(\lambda O_1)$ such that all families $\{\psi_j(\lambda)\}_{\lambda>0}$, $j = 1, \dots, d$, are asymptotically contained in $\mathcal{F}_{0,\iota}(O)$ if $O \supset \overline{O_1}$.

Let us briefly convince ourselves that each family $\{\psi_j(\lambda)\}_{\lambda>0}$ of a scaled multiplet satisfies the assumptions (i)–(iii) of Def. 5.2. Clearly, only condition (iii) need be checked, and one has

$$\sup_\lambda ||\beta_g(\psi_j(\lambda)) - \psi_j(\lambda)|| = \sup_\lambda ||\sum_{i=1}^d \psi_i(\lambda)(v_{[\rho]ij}(g) - \delta_{ij})|| \leq d \max_{i,j} |v_{[\rho]ij}(g) - \delta_{ij}|$$

where the last term tends to 0 if $g \to 1_G$ if G is a continuous group.

We remark that, in view of part (c) of Lemma 5.3, a similar criterion has been used recently by Morsella [16]. Part (c) of Lemma 5.3 also provides some insight into the basic mechanism which might cause charges to disappear in the scaling limit. To elaborate on that, we consider a scaled multiplet $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ for the charge $[\rho]$. Moreover, for $h \in L^1(\mathbb{R}^n)$ with compact support and $h \geq 0$, $\int d^n x\, h(x) = 1$, we define

$$\underline{\Phi}_\lambda^{(h,j)} = (\underline{\alpha}_h \psi_j)(\lambda)\,, \lambda > 0\,.$$

Now by Lemma 5.3 it follows that for the charge $[\rho]$ to be preserved in the scaling limit QFTGA of $\omega_{0,\iota}$, one must be able to choose a schaled multiplet and h in such a way that $||\pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}||$ comes arbitrarily close to 1. It could however happen that for all scaled multiplets and any choice of h one ends up with

$$||\pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}|| = 0\,,$$

which would also imply $\pi_{0,\iota}(\underline{\Phi}^{(h,j)}) = 0$ since $\Omega_{0,\iota}$ is separating for the local field algebras of the scaling limit QFTGA. We can interpret this as follows. The convolution of the scaled charge multiplets $\psi_j(\lambda)$ with respect to the scaled action of the translations, which produces elements $\underline{\Phi}^{(h,j)}$ in $\underline{\mathfrak{F}}$, results in an energy damping of the charged states that are obtained by applying the $\underline{\Phi}_\lambda^{(h,j}$ to the vacuum vector Ω. This energy damping scales inversely, that is, proportional to λ^{-1}, to the localization scale of the $\underline{\Phi}_\lambda^{(h,j)}$. Depending on the dynamics of the underlying QFSGS, it may happen that the amount of energy-momentum required to create the charged vectors $\psi_j(\lambda)\Omega$ from the vacuum in a small region of scale λ is typically larger than $\sim \lambda^{-1}$, e.g. of the type $\sim \lambda^{-q}$ with some $q > 1$. In this case, the energy damping leads to a "blotting out" of the charged contributions of $\underline{\Phi}_\lambda^{(h,j)}\Omega$, resulting in the vanishing of the norm of these vectors as λ approaches 0.

Concerning the question whether our criterion for preservance of charges is fulfilled in certain quantum field models, we note that [17] contains a result stating that the charges of the Majorana-Dirac field satisfy indeed this criterion in all scaling limit states.

Let us also sketch a physical picture of a possible — albeit quite hypothetical — mechanism of charge disappearance in the short distance scaling limit: This might occur if the dynamics of the underlying quantum field theory has the property that certain "compounds" of charges are dynamically more favourable than, e.g., certain single charges. That is to say, it may cost far less energy to create a compound of several charges at small scales than the single charges contained in the compound. In this case, the compound charges could survive the scaling limit (i.e. be preserved), while certain single charges disappear since their creation costs too much energy at small scales. The compound charges preserved in the scaling limit could then well be invariant under some normal subgroup of the gauge group of the underlying quantum field theory. In a sense, this mechanism is complementary to that of confinement at finite distances of charges which would be viewed as "free" charges in the short-distance scaling limit (asymptotic freedom) as in QCD. There, one expects that the colour charges correspond to charges which are present as superselection charges of a scaling limit quantum field theory (corresponding to field multiplets in $\mathcal{F}^{(0,\iota)}$, not in $\mathcal{F}_{0,\iota}$), while in the underlying quantum field theory, at finite scale, only colour-neutral compounds of the colour-charges appear. The sketched mechanism of charge disappearance in the scaling limit points at a strongly binding force between charges at extremely short distances, resulting in a sort of "asymptotic confinement".

Our criterion of charge preservance not only bars the situation of charge disappearance, but it even implies that the limits of $\pi_{0,\iota}(\underline{\Phi}^{(h,j)})$, $j = 1, \ldots, d$, as h tends to the δ-measure, yield charge multiplets corresponding to the charge $[\rho]$ with respect to their transformation behaviour under the scaling limit gauge group. This is the content of the following statement.

Proposition 5.5 *Suppose that the charge $[\rho]$ is preserved in the scaling limit QFTGA of $\omega_{0,\iota}$. Let $\{\psi_1(\lambda), \ldots, \psi_d(\lambda)\}_{\lambda>0}$ be a scaled multiplet for $[\rho]$ which is asymptotically contained in $\mathcal{F}_{0,\iota}(O)$, and let $\underline{\Phi}^{(h,j)}$ be defined as before with respect to the $\{\psi_j(\lambda)\}_{\lambda>0}$.*

Then the limit operators

$$\boldsymbol{\psi}_j = s\text{-}\lim_{\mu\to 0}\ \pi_{0,\iota}(\underline{\Phi}^{(h_\mu,j)}) \quad \text{and} \quad \boldsymbol{\psi}_j^* = s\text{-}\lim_{\mu\to 0}\ \pi_{0,\iota}(\underline{\Phi}^{(h_\mu,j)})^* \tag{5.3}$$

exist, are independent of h and are contained in $\mathcal{F}_{0,\iota}(\hat{O})$ whenever $\hat{O} \supset \overline{O}$. Furthermore, $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_d$ forms a multiplet transforming under the adjoint action of $U^\bullet_{0,\iota}(G^\bullet_{0,\iota})$ according to the irreducible, unitary representation $v_{[\rho]}$. More precisely, denoting by $G \ni g \mapsto g^\bullet \in G^\bullet_{0,\iota}$ the quotient map, there is a finite-dimensional, irreducible, unitary representation $v^\bullet_{[\rho]}$ of $G^\bullet_{0,\iota}$ so that $v^\bullet_{[\rho]}(g^\bullet) = v_{[\rho]}(g)$ for all $g \in G$ and

$$U^\bullet_{0,\iota}(g^\bullet)\boldsymbol{\psi}_j U^\bullet_{0,\iota}(g^\bullet)^* = \sum_{i=1}^{d} \boldsymbol{\psi}_i v^\bullet_{[\rho]ij}(g^\bullet)\,, \quad g^\bullet \in G^\bullet_{0,\iota}\,.$$

Proof. First we need to establish existence of the limit. Let h and $\tilde{h}$ be compactly supported, non-negative $L^1(\mathbb{R}^n)$-functions whose integrals are equal to 1. Choose any $\epsilon > 0$. Then one can find $\mu_0 > 0$ so that

$$\begin{aligned} &||(\ \pi_{0,\iota}(\underline{\Phi}^{(h_\mu,j)}) - \pi_{0,\iota}(\underline{\Phi}^{(\tilde{h}_{\tilde{\mu}},j)}\ )\Omega_{0,\iota}|| = \lim_\kappa ||(\underline{\Phi}^{(h_\mu,j)}_{\lambda_\kappa} - \underline{\Phi}^{(\tilde{h}_{\tilde{\mu}},j)}_{\lambda_\kappa})\Omega|| \\ &\leq\ \limsup_\kappa \Big(||(\underline{\Phi}^{(h_\mu,j)}_{\lambda_\kappa} - \psi_j(\lambda_\kappa))\Omega|| + ||(\underline{\Phi}^{(\tilde{h}_{\tilde{\mu}},j)}_{\lambda_\kappa} - \psi_j(\lambda_\kappa))\Omega|| \Big) < \epsilon \end{aligned}$$

if $0 < \mu, \tilde{\mu} < \mu_0$. This shows that $\pi_{0,\iota}(\underline{\Phi}^{(h_\mu,j)})\Omega_{0,\iota}$ is a Cauchy sequence in $\mu \to 0$ and hence has a limit in $\mathcal{H}_{0,\iota}$; it shows also that the limit is independent of h. Since $\Omega_{0,\iota}$ is separating for the local scaling limit field algebras and $||\underline{\Phi}^{(h_\mu,j)}||$ is bounded uniformly in μ, one can thus conclude that $\pi_{0,\iota}(\underline{\Phi}^{(h_\mu,j)})$ converges strongly to some $\boldsymbol{\psi}_j$ which is contained in $\mathcal{F}_{0,\iota}(\hat{O})$ if $\hat{O} \supset \overline{O}$. Similarly one argues that $\pi_{0,\iota}(\underline{\Phi}^{(h_\mu,j)})^*$ converges strongly to $\boldsymbol{\psi}_j^*$.

Next we demonstrate $\boldsymbol{\psi}_j^*\boldsymbol{\psi}_k = \delta_{jk}1$. To this end, we observe that for any $\underline{F} \in \underline{\mathfrak{F}}$ there holds the following chain of equations,

$$\begin{aligned}
&\langle \pi_{0,\iota}(\underline{F})\Omega_{0,\iota}, (\boldsymbol{\psi}_j^*\boldsymbol{\psi}_k - \delta_{jk}1)\Omega_{0,\iota}\rangle \\
&= \lim_{\mu\to 0}\lim_{\kappa} \, [\, \langle \underline{\Phi}_{\lambda_\kappa}^{(h_\mu,j)}\underline{F}_{\lambda_\kappa}\Omega, \underline{\Phi}_{\lambda_\kappa}^{(h_\mu,k)}\Omega\rangle - \delta_{jk}\langle \underline{F}_{\lambda_\kappa}\Omega,\Omega\rangle \,] \\
&= \lim_{\mu\to 0}\lim_{\kappa} \, [\, \langle \psi_j(\lambda_\kappa)\underline{F}_{\lambda_\kappa}\Omega, \psi_k(\lambda_\kappa)\Omega\rangle - \delta_{jk}\langle \underline{F}_{\lambda_\kappa}\Omega,\Omega\rangle \\
&\qquad\qquad + \langle (\underline{\Phi}_{\lambda_\kappa}^{(h_\mu,j)} - \psi_j(\lambda_\kappa))\underline{F}_{\lambda_\kappa}\Omega, \psi_k(\lambda_\kappa)\Omega\rangle \\
&\qquad\qquad + \langle \underline{\Phi}_{\lambda_\kappa}^{(h_\mu,j)}\Omega, (\underline{\Phi}_{\lambda_\kappa}^{(h_\mu,k)} - \psi_k(\lambda_\kappa))\Omega\rangle \,] \,.
\end{aligned}$$

The expression on the third to last line is equal to 0 since $\psi_j(\lambda)^*\psi_k(\lambda) = \delta_{jk}1$ by assumption, and the limits of the expressions on the last two lines vanish by the argument having led to the conclusion $(c) \Rightarrow (b)$ in the proof of Lemma 5.3. This proves $\boldsymbol{\psi}_j^*\boldsymbol{\psi}_k = \delta_{jk}1$ by the separating property of $\Omega_{0,\iota}$ for the local field algebras in the scaling limit.

The proof of $\sum_{j=1}^d \boldsymbol{\psi}_j\boldsymbol{\psi}_j^* = 1$ is completely analogous.

For the last part of the statement, we observe that

$$U_{0,\iota}(g)\boldsymbol{\psi}_j U_{0,\iota}(g)^* = \sum_{k=1}^d \boldsymbol{\psi}_k v_{[\rho]kj}(g)\,, \quad g \in G\,,$$

is simply a consequence of

$$\underline{\beta}_g(\underline{\Phi}^{(h,j)}) = \sum_{k=1}^d \underline{\Phi}^{(h,k)} v_{[\rho]kj}(g)\,, \quad g \in G\,;$$

this, in turn, can be seen from $\underline{\Phi}^{(h,j)} = \underline{\alpha}_h\psi_j$ and the commutativity of $\underline{\beta}_g$ and $\underline{\alpha}_x$.

On the other hand, from the definition of $N_{0,\iota}$ one obtains

$$\boldsymbol{\psi}_j\Omega_{0,\iota} = U_{0,\iota}(n)\boldsymbol{\psi}_j\Omega_{0,\iota} = \sum_{k=1}^d \boldsymbol{\psi}_k v_{[\rho]kj}(n)\Omega_{0,\iota}$$

for all $n \in N_{0,\iota}$, and multiplying by $\boldsymbol{\psi}_i^*$ from the left yields $\delta_{ij}\Omega_{0,\iota} = v_{[\rho]ij}(n)\Omega_{0,\iota}$ for all $n \in N_{0,\iota}$. This shows $v_{[\rho]}(n) = 1$ (the unit matrix) for all $n \in N_{0,\iota}$ and hence there is an irreducible, unitary representation $v_{[\rho]}^\bullet$ of G so that $v_{[\rho]}^\bullet(g^\bullet) = v_{[\rho]}(g)$ for all $g \in G$, proving the last part of the statement. $\square$

There is an obvious connection between the scaling limits of scaled multiplets for a charge $[\rho]$ and the scaling limits of endomorphisms induced by scaled multiplets in case that $[\rho]$ is preserved in a scaling limit state. While fairly immediate, we put the corresponding result on record here.

Proposition 5.6 *Let $\omega_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega)$ and let $[\rho] \in \mathrm{Sect}_{\mathrm{fin}}^{\mathrm{cov}}$ be a charge of the underlying QFSGS which is preserved in the scaling limit QFTGA of $\omega_{0,\iota}$. Moreover, let $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ be a scaled multiplet for $[\rho]$ asymptotically contained in $\mathfrak{F}_{0,\iota}(O)$ and let, with respect to this scaled multiplet, $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_d$ be defined as in* (5.3).

Then for each $\underline{A} \in \underline{\mathfrak{A}}$ the family $\{\rho(\underline{A})(\lambda)\}_{\lambda>0}$ defined by

$$\rho(\underline{A})(\lambda) = \sum_{j=1}^{d} \psi_j(\lambda)\underline{A}_\lambda \psi_j(\lambda)^*$$

is asymptotically contained in $\mathfrak{A}_{0,\iota}$. Furthermore, for each non-negative, compactly supported $h \in L^1(\mathbb{R}^n)$ with $\int d^n x\, h(x) = 1$ there holds

$$s\text{-}\lim_{\mu\to 0} \pi_{0,\iota}(\underline{\alpha}_{h_\mu}\rho(\underline{A})) = \sum_{j=1}^{d} \boldsymbol{\psi}_j \pi_{0,\iota}(\underline{A})\boldsymbol{\psi}_j^*\,, \quad \underline{A} \in \underline{\mathfrak{F}}\,; \tag{5.4}$$

and $\boldsymbol{\rho}$ defined by

$$\boldsymbol{\rho}(\boldsymbol{a}) = \sum_{j=1}^{d} \boldsymbol{\psi}_j \boldsymbol{a} \boldsymbol{\psi}_j^*\,, \quad \boldsymbol{a} \in \mathfrak{A}_{0,\iota}\,, \tag{5.5}$$

is a localized, transportable, irreducible endomorphism of $\mathfrak{A}_{0,\iota}$ which is moreover covariant and has finite statistics.

Proof. The asymptotic containment in $\mathfrak{A}_{0,\iota}$ of $\{\rho(\underline{A})(\lambda)\}_{\lambda>0}$ is simply a consequence of the asymptotic containment of each $\{\psi_j(\lambda)\}_{\lambda>0}$ in $\mathcal{F}_{0,\iota}(O)$ and the fact that $\rho(\underline{A})(\lambda) \in \mathcal{A}(\lambda(O_1 \cap O_2))$ for $\underline{A} \in \underline{\mathfrak{A}}(O_2)$, with the conventional assumption that $\psi_j(\lambda) \in \mathcal{F}(\lambda O_1)$. Owing to (5.5) and the properties of a multiplet, $\boldsymbol{\rho}$ is clearly a localized, irreducible endomorphism of $\mathfrak{A}_{0,\iota}$. The transportablity may seen as follows. According to the definition of preserved charge, there is for any double cone $O_\times$ different from O a scaled multiplet for $[\rho]$, $\{\tilde{\psi}_1(\lambda), \dots, \tilde{\psi}_d(\lambda)\}_{\lambda>0}$, which is asymptotically contained in $\mathcal{F}_{0,\iota}(O_\times)$. In the same way as the $\{\psi_j(\lambda)\}_{\lambda>0}$ lead to multiplet operators $\boldsymbol{\psi}_d$ in $\mathcal{F}_{0,\iota}(\hat{O})$ for all $\hat{O} \supset \overline{O}$, the $\{\tilde{\psi}_j(\lambda)\}_{\lambda>0}$ lead to multiplet operators $\tilde{\boldsymbol{\psi}}_j$ contained in $\mathcal{F}_{0,\iota}(\hat{O}_\times)$ for all $\hat{O}_\times \supset \overline{O}$. For the corresponding endomorphism $\tilde{\boldsymbol{\rho}}$ it then holds that $\boldsymbol{T}\tilde{\boldsymbol{\rho}}(\,.\,) = \boldsymbol{\rho}(\,.\,)\boldsymbol{T}$ with the unitary intertwiner $\boldsymbol{T} = \sum_{j=1}^{d} \boldsymbol{\psi}_j \tilde{\boldsymbol{\psi}}_j^*$. Now it is easy to see that the family $\{T(\lambda)\}_{\lambda>0}$ defined by $T(\lambda) = \sum_{j=1}^{d} \psi_j(\lambda)\tilde{\psi}_j^*$ is asymptotically contained in $\mathfrak{A}_{0,\iota}(O_*)$ for some double cone O_*, and by an argument by now familiar, $\boldsymbol{T} = s\text{-}\lim_{\mu\to 0} \pi_{0,\iota}(\underline{\alpha}_{h_\mu} T)$ showing that $\boldsymbol{T}$ is contained in $\mathcal{A}_{0,\iota}(\hat{O}_*)$ for $\hat{O}_* \supset \overline{O_*}$. Covarivance follows from a general argument: Given a multiplet $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_d$, it holds that $\boldsymbol{\rho}(U\boldsymbol{a}U^*) = W\boldsymbol{\rho}(a)W^*$ for each unitary U with $W = \sum_{j=1}^{d} \boldsymbol{\psi}_j U \boldsymbol{\psi}_j^*$ which is itself unitary. That $\boldsymbol{\rho}$ has finite statistics follows from the finiteness of the dimension d of the multiplet. $\square$

The last result presented in this section concerns the preservance of the conjugate charge of a preserved charge. To this end, let us assume for the remainder of this section that the underlying QFSGS fulfills also the condition of geometric modular action as formulated in (QFTGA.9) in Sec. 3. (We note that this can be deduced already if a similar form of geometric modular action is initially only assumed to hold for the underlying observable quantum system provided it fulfills some mild additional conditions. We refer to [9, 10, 14] for discussion of this issue.) In this case, let $\psi_1, \dots, \psi_d$ be a multiplet for the charge $[\rho] \in \mathrm{Sect}_{\mathrm{fin}}^{\mathrm{cov}}$, with all ψ_j contained in $\mathcal{F}(O)$ for some $O \in \mathcal{K}$, and assume that W

is a wedge region containing O. Let J_W denote the Tomita-Takesaki modular conjugation associated with $\mathcal{F}(W)$ and the vacuum vector Ω. Then one can take an arbitrary multiplet $\psi'_1, \dots, \psi'_d$ for $[\rho]$ with all $\psi'_j \in \mathcal{F}(r_W O)$, and define a new multiplet of operators $\overline{\psi}_j \in \mathcal{F}(O)$, $j = 1, \dots, d$, by

$$\overline{\psi}_j = J_W V \psi'_j V^* J_W$$

where V is the "twist" operator defined in (3.2). It is easy to check that the $\overline{\psi}_j$ indeed form a multiplet, i.e. $\sum_{j=1}^d \overline{\psi}_j \overline{\psi}_j^* = 1$ and $\overline{\psi}_j^* \overline{\psi}_k = \delta_{jk} 1$; however, since J_W is antilinear, this multiplet transforms under the gauge group action according to the conjugate representation $\overline{v}_{[\rho]}$ of $v_{[\rho]}$,

$$U(g)\overline{\psi}_j U(g)^* = \sum_{i=1}^d \overline{\psi}_i \overline{v}_{[\rho]ij}(g) \,, \quad g \in G \,,$$

if $\psi_1, \dots, \psi_d$ transforms under the gauge group according to $v_{[\rho]}$. This indicates that $\overline{\psi}_1, \dots, \overline{\psi}_d$ is a multiplet of the conjugate sector $[\overline{\rho}]$ of $[\rho]$. Indeed, writing

$$\rho(A) = \sum_{j=1}^d \psi_j A \psi_j^* \,, \quad \overline{\rho}(A) = \sum_{j=1}^d \overline{\psi}_j A \overline{\psi}_j^* \,, \quad R = \sum_{j=1}^d \overline{\psi}_j \psi_j \,, \quad \overline{R} = \sum_{j=1}^d \psi_j \overline{\psi}_j \,,$$

one can easily check that R and $\overline{R}$ are isometries in $\mathcal{A}(O)$ and moreover, there holds

$$\overline{\rho}(\rho(A))R = RA \quad \text{and} \quad \rho(\overline{\rho}(A))\overline{R} = \overline{R}A \,, \quad A \in \mathfrak{A} \,.$$

(This can actually also be deduced from a rather more general argument of [9].)

Equipped with these observations, we can now state the result.

Theorem 5.7 *Suppose that the underlying QFSGS fulfills the condition of geometric modular action (QFTGA.9), and let $\omega_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega)$ be one of its scaling limit states. Then a charge $[\rho] \in \mathrm{Sect}^{\mathrm{cov}}_{\mathrm{fin}}$ is preserved in the scaling limit state $\omega_{0,\iota}$ if and only if also the conjugate charge $[\overline{\rho}]$ is preserved.*

Proof. Assume that $[\rho]$ is preserved in the scaling limit state $\omega_{0,\iota}$ and let $O \in \mathcal{K}$. Then for $O_1 \in \mathcal{K}$ with $\overline{O_1} \subset O$ there is a scaled multiplet $\{\psi'_1(\lambda), \dots, \psi'_d(\lambda)\}_{\lambda > 0}$ for $[\rho]$, contained in $\mathcal{F}(\lambda r_W O_1)$ and asymptotically contained in $\mathcal{F}_{0,\iota}(O_1)$. Let $\overline{\psi}_j$ be defined by

$$\overline{\psi}_j(\lambda) = J_W V \psi'_j(\lambda) V^* J_W$$

where V is the "twist" operator (cf. eq. (3.2)) and J_W is the modular conjugation associated with $\mathcal{F}(W)$ and the vacuum vector Ω. Then $\{\overline{\psi}_1(\lambda), \dots, \overline{\psi}_d(\lambda)\}_{\lambda > 0}$ is a scaled multiplet for the conjugate charge $[\overline{\rho}]$ and each $\overline{\psi}_j(\lambda)$ is contained in $\mathcal{F}(\lambda O_1)$. Moreover, for compactly supported $h \in L^1(\mathbb{R}^n)$ it holds that

$$(\underline{\alpha}_h \overline{\psi}_j)(\lambda) = J_W V (\underline{\alpha}_{h \circ r_W} \psi'_j)(\lambda) V^* J$$

and this shows that the $\{\overline{\psi}_j(\lambda)\}_{\lambda > 0}$ are asymptotically contained in $\mathcal{F}_{0,\iota}(O)$. $\square$

Remarks. (i) Note that under the conditions of Thm. 5.7 one also obtains asymptotic scaling limit versions of the isometries which intertwine $\boldsymbol{\rho}$ and $\overline{\boldsymbol{\rho}}$. More precisely, suppose that a charge $[\rho]$ is preserved in the scaling limit state $\omega_{0,\iota}$, and let $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_d$ be a corresponding multiplet contained in $\mathcal{F}_{0,\iota}(O)$ induced by a scaled multiplet $\{\underline{\psi_1}(\lambda), \dots, \underline{\psi_d}(\lambda)\}_{\lambda>0}$. As the previous Theorem shows, there is then a conjugate multiplet $\overline{\boldsymbol{\psi}}_1, \dots, \overline{\boldsymbol{\psi}}_d$ in $\mathcal{F}_{0,\iota}(O)$ induced by a scaled multiplet $\{\underline{\overline{\psi}_1}(\lambda), \dots, \underline{\overline{\psi}_d}(\lambda)\}_{\lambda>0}$, and it is straightforward to show that $\boldsymbol{R} = \sum_{j=1}^d \overline{\boldsymbol{\psi}}_j \boldsymbol{\psi}_j$ and $\overline{\boldsymbol{R}} = \sum_{j=1}^d \boldsymbol{\psi}_j \overline{\boldsymbol{\psi}}_j$ are given by

$$\boldsymbol{R} = s\text{-}\lim_{\mu\to 0} \sum_{j=1}^d \pi_{0,\iota}(\underline{\alpha}_{h_\mu} \underline{\overline{\psi}_j \psi_j}) \quad \text{and} \quad \overline{\boldsymbol{R}} = s\text{-}\lim_{\mu\to 0} \sum_{j=1}^d \pi_{0,\iota}(\underline{\alpha}_{h_\mu} \underline{\psi_j \overline{\psi}_j})$$

where $h \in L^1(\mathbb{R}^n)$ is non-negative, with compact support and $\int d^n x\, h(x) = 1$. Using this, one deduces

$$\overline{\boldsymbol{\rho}}(\boldsymbol{\rho}(\boldsymbol{a}))\boldsymbol{R} = \boldsymbol{R}\boldsymbol{a} \quad \text{and} \quad \boldsymbol{\rho}(\overline{\boldsymbol{\rho}}(\boldsymbol{a}))\overline{\boldsymbol{R}} = \overline{\boldsymbol{R}}\boldsymbol{a}\,, \quad \boldsymbol{a} \in \mathfrak{A}_{0,\iota}\,,$$

where $\boldsymbol{\rho}$ and $\overline{\boldsymbol{\rho}}$ relate to the $\boldsymbol{\psi}_j$ and $\overline{\boldsymbol{\psi}}_j$, respectively, as in (5.5).

(ii) Note that we have not assumed that the underlying QFSGS is Lorentz covariant, i.e. we have not imposed (QFTGA.6). If we make this assumption in addition to (QFTGA.9), and define the scaling algebra $\underline{\mathfrak{F}}$ according to the Convention stated below (QFTGA.9), then we obtain the following: Let $J_{W\,0,\iota}$ and $V_{0,\iota}$ denote the analogous objects to J_W and V in the scaling limit theory of $\omega_{0,\iota}$, then a conjugate charge multiplet $\overline{\boldsymbol{\psi}}_1, \dots, \overline{\boldsymbol{\psi}}_d$ to $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_d$ is obtained by

$$\overline{\boldsymbol{\psi}}_j = J_{W\,0,\iota} V_{0,\iota} \boldsymbol{\psi}'_j V^*_{0,\iota} J_{W\,0,\iota}$$

whenever $\boldsymbol{\psi}'_1, \dots, \boldsymbol{\psi}'_d$ is a multiplet equivalent to $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_d$ localized in $r_W O$.

6 On Equivalence of Local and Global Intertwiners

In the present section we will address the question of equivalence of local and global intertwiners of superselection sectors. We shall extend an argument of Roberts [19] who considered the setting of dilation covariant quantum field theories, showing that the preservance of all charges in some scaling limit theories is, together with the assumption that the local field algebras $\mathcal{F}(O)$ are factors, sufficient for the equivalence of local and global intertwiners. Our main technical result is stated in the following Lemma.

Lemma 6.1 *Let $[\rho] \in \mathrm{Sect}^{\mathrm{cov}}_{\mathrm{fin}}$ be a superselection sector of the underlying QFSGS, let $O \in \mathcal{K}$, and suppose that there are (i) a scaling limit state $\omega_{0,\iota} \in \mathrm{SL}^{\mathcal{F}}(\omega)$, (ii) a scaled multiplet $\{\psi_1(\lambda), \dots, \psi_d(\lambda)\}_{\lambda>0}$ for $[\rho]$ with $\psi_j(\lambda) \in \mathcal{F}(\lambda O)$, (iii) some compactly supported, non-negative $h \in L^1(\mathbb{R}^n)$, such that*

$$||\pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}|| > 0\,, \quad j = 1, \dots, d\,,$$

where $\underline{\Phi}^{(h,j)}_\lambda = (\underline{\alpha}_h \psi_j)(\lambda)$. Then for all unitaries $U \in \mathcal{A}(O)' \cap \mathcal{F}(O)$ and all multiplets $\tilde{\psi}_1, \dots, \tilde{\psi}_d \in \mathcal{F}(O)$ for $[\rho]$ $(O \in \mathcal{K})$ there holds

$$\omega(\tilde{\psi}_j^* U^* \tilde{\psi}_k U) = \delta_{jk}\,.$$

Proof. First we note that $||\pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}|| > 0$ for any of the $j = 1, \ldots, d$ implies that the $\pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}$, $j = 1, \ldots, d$, are linearly independent. To see this, note that the contrary assumption of linear dependence implies that there is an invertible $d \times d$ matrix $(u_{j\ell})$ so that $\sum_j \pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}u_{j\ell} = 0$ for some ℓ. But this implies

$$0 = U_{0,\iota}(g)\sum_j \pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}u_{j\ell} = \sum_{j,k} \pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}v_{[\rho]kj}(g)u_{j\ell}$$

for all $g \in G$ and hence, since $v_{[\rho]}$ is irreducible, $\pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota} = 0$ for all j.

We further observe that it constitutes no restriction of generality to prove the statement of the theorem only for $O \in \mathcal{K}$ which contain the origin $0 \in \mathbb{R}^n$ in their spacelike boundary (i.e. the origin is contained both in the boundary of O and in the boundary of its spacelike complement) since the underlying QFSGS is translation covariant. Thus we continue to prove the statement for an arbitrary O of this type.

We begin by noting that from our observation above, the $\pi_{0,\iota}(\underline{\Phi}^{(h,j)})\Omega_{0,\iota}$, $j = 1, \ldots, d$, span a d-dimensional subspace of $\mathcal{H}_{0,\iota}$. Now let $W \supset O$ be a wedge region containing the origin in its spacelike boundary. Then let W' be the wedge which is the causal complement of W, and let W'_h be a copy of W' shifted by some suitable spacelike vector into the interior of W' such that W'_h lies in the causal complement of $O + \operatorname{supp} h$. By a standard Reeh-Schlieder argument $\mathcal{F}_{0,\iota}(W'_h)\Omega_{0,\iota}$ is dense in $\mathcal{H}_{0,\iota}$ and hence, choosing some $\epsilon > 0$ arbitrarily, there will be some double cone $\hat{O} \subset W'_h$ and $\underline{F}^{(1)}, \ldots, \underline{F}^{(d)} \in \underline{\mathfrak{F}}(\hat{O})$ such that

$$|\langle \pi_{0,\iota}(\underline{F}^{(j)})^*\Omega_{0,\iota}, \pi_{0,\iota}(\underline{\Phi}^{(h,k)})\Omega_{0,\iota}\rangle - \delta_{jk}| = |\omega_{0,\iota}(\underline{F}^{(j)}\underline{\Phi}^{(h,k)}) - \delta_{jk}| < \epsilon\,.$$

Now let $(\lambda_\kappa)_{k\in K}$ be a subnet of the positive reals, converging to 0, with $\omega_{0,\iota} = \lim_\kappa \underline{\omega}_{\lambda_\kappa}$ on $\underline{\mathfrak{F}}$. Since $\underline{F}^{(j)}_{\lambda_\kappa}\underline{\Phi}^{(h,k)}_{\lambda_\kappa}$ converges weakly to a multiple of 1 owing to $\bigcap_{O\ni 0}\mathcal{F}(O) = \mathbb{C}1$ (see [19]), we obtain

$$\omega_{0,\iota}(\underline{F}^{(j)}\underline{\Phi}^{(h,k)}_{\lambda_\kappa}) = \lim_\kappa \omega(\underline{F}^{(j)}_{\lambda_\kappa}\underline{\Phi}^{(h,k)}_{\lambda_\kappa}) = \lim_\kappa \omega'(\underline{F}^{(j)}_{\lambda_\kappa}\underline{\Phi}^{(h,k)}_{\lambda_\kappa})$$

for each locally normal state ω' on $\mathfrak{F}$, and this implies

$$|\lim_\kappa \omega'(\underline{F}^{(j)}_{\lambda_\kappa}\underline{\Phi}^{(h,k)}_{\lambda_\kappa}) - \delta_{jk}| < \epsilon$$

whenever ω' is locally normal. On the other hand, since $||(\alpha_{\lambda x}(U) - U)\Omega|| \to 0$ as $\lambda \to 0$ uniformly for x ranging over compact sets, it follows that

$$\begin{aligned}
&\omega(U^*\underline{F}^{(j)}_{\lambda_\kappa}\underline{\Phi}^{(h,k)}_{\lambda_\kappa}U)\\
&= \int d^nx\, h(x)\omega(\alpha_{\lambda_\kappa x}(U^*)\underline{F}^{(j)}_{\lambda_\kappa}\alpha_{\lambda_\kappa x}(\psi_k(\lambda_\kappa))U)) + o(\lambda_\kappa)\\
&= \int d^nx\, h(x)\omega(U^*\alpha_{-\lambda_\kappa x}(\underline{F}^{(j)})_{\lambda_\kappa}\psi_k(\lambda_\kappa)\alpha_{-\lambda_\kappa x}(U)) + o(\lambda_\kappa)\,,
\end{aligned}$$

where $o(\lambda)$ tends to 0 for $\lambda \to 0$, and we have used invarince of the vacuum state ω under the action of the translations α_x. We have also inserted the definition of the $\underline{\Phi}^{(h,k)}$, so that the scaled multiplets $\psi_k(\lambda)$ appear here.

Next we write $\psi_j(\lambda = 1) = \psi_j$, and we notice that $\psi_j(\lambda) = T_\lambda \psi_j$ where $T_\lambda = \sum_{j=1}^d \psi_j(\lambda)\psi_j^*$ is contained in $\mathcal{A}(O)$, and thus commutes with $U \in \mathcal{A}(O)' \cap \mathcal{F}(O)$. We note also that for every $B \in \mathfrak{F}$ we have, denoting by V the "twist" operator of (3.2),

$$\begin{aligned} & \omega(U^* \alpha_{-\lambda_\kappa x}(\underline{F}^{(j)}_{\lambda_\kappa})B) = \omega(VU^*V^*V\alpha_{-\lambda_\kappa x}(\underline{F}^{(j)}_{\lambda_\kappa})V^*VBV^*) \\ = \; & \omega(V\alpha_{-\lambda_\kappa x}(\underline{F}^{(j)}_{\lambda_\kappa})V^*(VU^*V^*)VBV^*) = \omega(\alpha_{-\lambda_\kappa x}(\underline{F}^{(j)}_{\lambda_\kappa})U^*B) \end{aligned}$$

for $\lambda_\kappa \leq 1$ and $x \in \operatorname{supp} h$ since then $\alpha_{-\lambda_\kappa x}(\underline{F}^{(j)}_{\lambda_\kappa}) \in \mathcal{F}(W')$ and $U^* \in \mathcal{A}(O)' \cap \mathcal{F}(O) \subset \mathcal{F}(W)$. Hence we get for $\lambda_\kappa \leq 1$,

$$\begin{aligned} & \int d^n x \, h(x) \omega(U^* \alpha_{-\lambda_\kappa x}(\underline{F}^{(j)})_{\lambda_\kappa} \psi_k(\lambda_\kappa)\alpha_{-\lambda_\kappa x}(U)) \\ = \; & \int d^n x \, h(x) \omega(\alpha_{-\lambda_\kappa x}(\underline{F}^{(j)}_{\lambda_\kappa}) T_{\lambda_\kappa}(\sum_{i=1}^d \psi_i\psi_i^*)U^*\psi_k \alpha_{-\lambda_\kappa x}(U)) \\ = \; & \sum_{i=1}^d \int d^n x \, \omega(\alpha_{-\lambda_\kappa x}(\underline{F}^{(j)}_{\lambda_\kappa}) T_{\lambda_\kappa} \psi_i \alpha_{-\lambda_\kappa x}(\psi_i^* U^* \psi_k U)) + p(\lambda_\kappa) \\ = \; & \sum_{i=1}^d \omega(\underline{F}^{(j)}_{\lambda_\kappa} \underline{\Phi}^{(h,i)}_{\lambda_\kappa} \psi_i^* U^* \psi_k U) + p(\lambda_\kappa) \end{aligned}$$

with some function $p(\lambda)$ tending to 0 as $\lambda \to 0$, where we used that

$$\lim_{\lambda \to 0} ||(\psi_i^* U^* \psi_k \alpha_{-\lambda x}(U) - \alpha_{-\lambda x}(\psi_i^* U^* \psi_k U))\Omega|| = 0$$

uniformly for x ranging over compact sets. Also we used the translational invariance of ω again. Summing up these findings we have for $\lambda_\kappa \leq 1$,

$$\omega(U^* \underline{F}^{(j)}_{\lambda_\kappa} \underline{\Phi}^{(h,k)}_{\lambda_\kappa} U) = \sum_{i=1}^d \omega(\underline{F}^{(j)}_{\lambda_\kappa} \underline{\Phi}^{(h,i)}_{\lambda_\kappa} \psi_i^* U^* \psi_k U) + o(\lambda_\kappa) + p(\lambda_\kappa) \,.$$

Making now use of the fact that for all normal states ω' it holds that

$$\lim_\kappa |\omega'(\underline{F}^{(j)}_{\lambda_\kappa} \underline{\Phi}^{(h,i)}_{\lambda_\kappa}) - \delta_{ji}| < \epsilon \,,$$

the previous equation yields, upon taking the limit over κ,

$$|\omega(\psi_j^* U^* \psi_k U) - \delta_{jk}| < (d+1)\epsilon \,.$$

Here $\epsilon > 0$ was arbitrary, and hence we conclude that

$$\omega(\psi_j^* U^* \psi_k U) = \delta_{jk}$$

holds for all unitary $U \in \mathcal{A}(O)' \cap \mathcal{F}(O)$ and the special multiplet $\psi_j = \psi_j(\lambda = 1)$. However, given any other multiplet $\tilde{\psi}_j$ in $\mathcal{F}(O)$ for the charge $[\rho]$, there is the unitary $T = \sum_{j=1}^d \tilde{\psi}_j \psi_j^*$ in $\mathcal{A}(O)$ so that $\tilde{\psi}_j = T\psi_j$, and thus we obtain, for each unitary $U \in \mathcal{A}(O)' \cap \mathcal{F}(O)$,

$$\omega(\tilde{\psi}_j^* U^* \tilde{\psi}_k U) = \omega(\psi_j^* T^* U^* T \psi_k U) = \omega(\psi_j^* U^* \psi_k U) = \delta_{jk}$$

since U and T commute. $\square$

Now we make use of the following result which has been proved in [19] (using also [7]): If, for some $O \in \mathcal{K}$, there holds

$$\omega(\tilde{\psi}_j^* U^* \tilde{\psi}_k U) = \delta_{jk}$$

for all charge multiplets $\tilde{\psi}_j$ (of all superselection sectors) contained in $\mathcal{F}(O)$ and for all unitaries U contained in $\mathcal{A}(O)' \cap \mathcal{F}(O)$, then

$$\mathcal{A}(O)' \cap \mathcal{F}(O) = \mathcal{F}(O)' \cap \mathcal{F}(O) \, .$$

If moreover the local field algebras of the underlying QFSGS are factors, i.e. if

$$\mathcal{F}(O) \cap \mathcal{F}(O)' = \mathbb{C}1 \, , \quad O \in \mathcal{K} \, , \tag{6.1}$$

then equivalence of local and global intertwiners ensues: Given $[\rho]$ and $[\rho']$ in $\mathrm{Sect}_{\mathrm{fin}}^{\mathrm{cov}}$ it holds that

$$\mathcal{I}(\rho, \rho')_O = \mathcal{I}(\rho, \rho') \quad \text{for } \rho, \rho' \text{ localized in } O \, . \tag{6.2}$$

(Cf. Sec. 4 for the definition of $\mathcal{I}(\rho, \rho')_O$ and $\mathcal{I}(\rho, \rho')$.)

Corollary 6.2 *Suppose that all local field algebras of the underlying QFSGS are factors, i.e. that* (6.1) *holds for all* $O \in \mathcal{K}$*. Moreover, suppose that for each charge* $[\rho] \in \mathrm{Sect}_{\mathrm{fin}}^{\mathrm{cov}}$ *there is some scaling limit state* $\omega_{0,\iota} \in \mathrm{SL}^{\mathfrak{F}}(\omega)$ *(which may depend on* $[\rho]$*) such that* $[\rho]$ *is preserved in that scaling limit state. Then in the underlying QFSGS there holds the equivalence of local and global intertwiners* (6.2)*.*

The factorial property of the local field algebras has been checked in free field models. Assuming that this is a general feature of quantum field theories, the assertion of the Corollary shows that part of the charge superselection structure is determined entirely locally if all charges are preserved in suitable scaling limit states; in other words, if the charges are, in this (somewhat generalized) sense, ultraviolet stable. For further discussion as to how much else of the superselection structure may be determined locally, we refer to [19].

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